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AMERICAN MATHEMATICAL SOCIETY

A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE

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THE THIRTY-FIFTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY

The thirty-fifth summer meeting and thirteenth colloquium of the Society were held at the University of Colorado, Boulder, Colorado, from Tuesday to Friday, August 27–30, 1929, preceded by the summer meeting of the Mathematical Association of America.

The four colloquium lectures by Professor R. L. Moore, each of an hour and one-quarter, were delivered on Tuesday and Thursday afternoons and Thursday and Friday mornings, with ninety-one members in attendance; a report of these lectures, by Professor G. T. Whyburn, appears in the present issue of this Bulletin. On Thursday morning the Society held sectional sessions for the reading of papers, a Section of Analysis, and a Section of Algebra. General sessions were held on Tuesday afternoon and Friday morning. The retiring address of President Snyder, entitled *The problem of the cubic variety* in S_4, which has already appeared on pages 607–642 of the September-October issue of this Bulletin, was delivered on Thursday afternoon. In place of the joint dinner of the Society and the Association, there was an excursion to University Camp, where a dinner was served, after which there were informal speeches, Professor A. J. Kempner presiding.

This is the farthest west a summer meeting has been held since 1915, and in every particular it justified the high hopes of those interested in the expansion of the area of the Society's activities. Boulder is picturesquely situated, the University's ample facilities were placed unreservedly and efficiently at the disposal of the visitors, the attendance was good, and the scientific standard of the meetings was high. The scientists came from twenty-eight states and Canada, representing nearly all sections of the country. Especially memorable among mathematical outings were the dinner at University

Camp in the mountains on Tuesday evening and the day trip on Wednesday to the continental divide at Fall River Pass, nearly twelve thousand feet above sea level.

At the concluding session, by a rising vote, resolutions of thanks were passed to the President and Regents of the University, to the members of the department of mathematics, to those who furnished transportation into the mountains, and to the Boulder Country Club.

The college fraternity houses were opened for lodging the visitors, and meals were served at a cafeteria near the campus, all at modest rates.

The attendance included the following one hundred nine members of the Society:

Bacon, S. R. Benedict, William James Berry, Brink, Bruce, H. E. Buchanan, Burnam, Bussey, Cairns, C. C. Camp, A. G. Clark, Abraham Cohen, Teresa Cohen, Colyer, L. P. Copeland, C. F. Craig, Curtiss, D. R. Davis, H. T. Davis, Dean, Denton, Dines, Dodd, Edmonson, Epsteen, Erikson, H. P. Evans, Fitterer, L. R. Ford, W. B. Ford, Fry, Gaba, J. L. Gibson, Gouwens, Griffin, Griffiths, W. L. Hart, Hazard, E. R. Hedrick, Hennel, Herr, Hicks, Hildebrandt, J. C. Hughes, Hurst, C. A. Hutchinson, Ingraham, Dunham Jackson, Kempner, Kendall, Kindle, Kuhn, Latimer, D. N. Lehmer, Lester, A. J. Lewis, Lubben, McFarland, McMaster, Mickelson, W. E. Milne, U. G. Mitchell, Mossman, R. L. Moore, W. K. Nelson, Newton, Nyswander, H. L. Olson, F. W. Owens, H. B. Owens, Pierce, Pierpont, M. B. Porter, Rambo, Raynor, Rawles, Rechard, Remick, R. G. D. Richardson, Richert, H. L. Rietz, J. H. Roberts, T. G. Rodgers, Roever, Rutt, Shewhart, Sisam, Skinner, Smail, C. E. Smith, E. R. Smith, G. W. Smith, R. G. Smith, Virgil Snyder, Spooner, Eugene Stephens, Stouffer, Tappan, Tripp, Tyler, L. E. Ward, Weida, Weiss, Wester, J. J. Wheeler, M. B. White, G. T. Whyburn, Winger, Zippin.

The secretary announced the entrance of the following eighty persons to membership in the Society:

Mr. James Edward Barker, University of California;
Mr. Amos Hale Black, West Virginia University;
Miss Dorothy Stahl Brady, New York University;
Mr. Frederick James Brand, University of Washington;
Mr. Joseph Hobart Bushey, University of Michigan;
Mr. Lloyd Cline Christianson, Texas Technological College;
Miss Nancy Cole, Radcliffe College;
Mr. Hollis R. Cooley; New York University;
Mr. Prescott Durand Crout, Milwaukee, Wisconsin;
Mr. Albert Eldred Currier, Harvard University;
Mr. Edward Hutchins Cutler, Harvard University;

Dr. Dewey Colfax Duncan, University of California;
Mr. John Dyer Elder, California Institute of Technology;
Mr. Orin James Farrell, Pennsylvania State College;
Mr. Arne Fisher, Western Union Telephone and Telegraph Company, New York City;
Miss Claire Beatrice Fisher, New York University;
Miss Elizabeth Agnes Flood, University of Montana;
Mr. David Herman Frank, George Washington High School, New York City;
Mr. Kenneth Gary Fuller, Northwestern University;
Mr. Michael Goldberg, Bureau of Ordnance, United States Navy;
Mr. Jacob Goodman, Rutgers University; Mr. Edward Kenneth Haviland, Harvard University;
Mr. William Jonathan Hazard, University of Colorado;
Mr. Charles Harold Helliwell, New York University;
Mr. Magnus Rudolph Hestenes, University of Nebraska;
Mr. John James Luett Hinrichsen, Iowa State College;
Miss Catherine Hood, Colorado College;
Professor Borden P. Hoover, Carnegie Institute of Technology;
Professor Frederick W. John, New York University;
Professor Francis Edgar Johnston, George Washington University;
Dr. Franklin Wesley Kokomoor, University of Florida;
Professor Michael Kovalenko, Swarthmore College;
Mr. Lonnie Langston, Texas Technological College;
Professor Victor F. Lenzen, University of California;
Mr. Arthur James Lewis, University of Denver;
Mr. Chin-Nien Liu, Nan Kai University, Tienstin, China;
Professor Hilton Frank Lusk, College of the Pacific;
Dr. Albert Joseph McConnell, Trinity College, Dublin, Ireland;
Miss Dorothy McCoy, State University of Iowa;
Mr William Milton Maiden, New York University;
Mr. Monroe Harnish Martin, Johns Hopkins University;
Professor Wendell Earle Mason, University of California at Los Angeles;
Mr. Frederic Howell Miller, Columbia University;
Dr. Alfred King Mitchell, Johns Hopkins University;
Mr. Morris Monsky, Mutual Life Insurance Company, New York City;
Dr. Walter Lee Moore, University of Illinois;
Mr. William Richard Morgans, California Institute of Technology;
Dr. William Kelso Morrill, Johns Hopkins University;
Mr. David Moskovitz, Carnegie Institute of Technology;
Mr. J. Herbert Moss, New York University;
Professor Walter Kenneth Nelson, University of Colorado;
Mr. James Edward O'Connell, Morris Preparatory School, Little Rock, Arkansas;
Professor Edwin Glenn Olds, Carnegie Institute of Technology;
Miss Emma Julia Olson, Northwestern University;
Mr. Charles Kenneth Payne, New York University;
Dr. Joseph William Peters, University of Illinois;

Professor Saul Pollock, Coast Artillery Corps, Fort Collins, Colorado;
Mr. Griffith Baley Price, Harvard University;
Professor A. W. Richeson, University of Maryland;
Professor Ramon Salas-Edwards, University of Chile, Santiago, Chile;
Mr. William Francis Shaw, vice-president, American Rio Grande Land and Irrigation Company;
Miss C. Grace Shover, Ohio State University;
Mr. Clifford Edward Smith, Fresno State College;
Professor Turner L. Smith, Carnegie Institute of Technology;
Mr. Clyde Allen Spicer, Johns Hopkins University;
Dr. Lee Houghton Swinford, University of California;
Professor Herbert Tate, McGill University;
Mr. Henry Peter Thielman, Ohio State University;
Mr. William Rae Thompson, Yale University;
Professor Arthur Tilley, New York University;
Professor Louis Russell Weber, Friend's University, Wichita;
Mr. Edward Henry Wells, Princeton University;
Professor George Warren White, College of the Pacific, Stockton, California;
Professor Edwin A. Whitman, Carnegie Institute of Technology;
Mr. Hassler Whitney, Yale University;
Mr. Charles Robert Wilson, Rutgers University;
Mr. Herbert Paul Wirth, College of the City of New York;
Mr. Kamcheung Woo, University of California;
Professor Leon P. Woods, Northeastern State Teachers College, Tahlequah, Oklahoma;
Dr. Minoru Yoshida, Sendai, Japan.

Eight applications for membership were received.

The Secretary also announced the following election to sustaining membership:

Members of the Department of Mathematics in Brown University.

The following appointments were announced: to represent the Society at the inauguration of President Clarence A. Barbour at Brown University, Professor G. D. Birkhoff; to represent the Society at the fiftieth anniversary celebration of the American Society of Mechanical Engineers, Dr. G. A. Campbell and Professor E. V. Huntington; as Committee on Arrangements for the summer meeting of 1930, at Brown University, Professors C. R. Adams (chairman), J. L. Coolidge, C. H. Currier, R. G. D. Richardson and H. W. Tyler.

It was announced that for 1929–30 the National Academy of Sciences had allotted to the Society from funds donated

by the General Education Board the sum of $4500 to be distributed as follows: Bulletin, $500; Transactions, $1500; American Journal of Mathematics, $1500; Colloquium, $1000.

The list of nominations for officers and other members of the Council was adopted and ordered printed on the ballot.

Titles and abstracts of the papers read at this meeting follow below. The papers numbered 1 to 7 were read at the general session on Tuesday afternoon, President Hedrick presiding; the papers numbered 8 to 23 were read before the Section of Analysis, ex-President Snyder and Professor Curtiss presiding; those numbered 24 to 43 before the Section of Algebra, Vice-President Rietz and Professor Cohen presiding; those numbered 44 to 60 at the general session on Friday morning, President Hedrick and Professor Tyler presiding. Dr. W. D. Baten was introduced by Professor J. A. Shohat and Professor Uspensky by the Secretary. The papers of Ayres, Campbell, Carson, Garver, Hill, Kempner, Langer, LaPaz, Lehmer, Lubben, McDonnell, Marden, Poritsky, Rainich, Robinson, Roth, Rowe, Rutledge, Simmons, Thielman, Vandiver, Ward (second paper), Wilder and Wilson were read by title.

1. Professor J. A. Nyswander: *On the solutions of linear differential equations.*

This paper gives a simple and direct derivation of the form of the solution of a general system of linear differential equations in the vicinity of a point $t=0$ which is an isolated uniform singularity for the coefficients of the system of differential equations. Solutions of the system in the vicinity of a regular point $t=t_0$ are determined so that after a circuit about the point $t=0$, certain specified changes occur in the solutions. In order that these changes shall result from analytic continuation about the singular point $t=0$ certain relations among the coefficients of the solutions must exist. That these relations can be satisfied is proved by differentiating a set of identities obtained from the characteristic determinant. The form of the solutions is established under entirely general hypotheses regarding the roots of the characteristic equation, without recourse to the theory of transformations.

2. Professor C. H. Sisam: *On three-dimensional varieties with six generations by right lines.*

In this paper, the author shows that the varieties mentioned in the title are cubic hypersurfaces in space of four dimensions.

3. Professor G. Y. Rainich: *Decomposition of a symmetric tensor in space-time.*

It is proved in this paper that in four-dimensional pseudo-euclidean space a non-singular symmetric tensor of rank two can be split up in only one way into two parts which possess respectively the formal properties of the hydrodynamical tensor (tensor of matter) and the electromagnetic stress-energy tensor. Applying this result and results previously established to the contracted Riemann tensor, it follows that curvature of space-time determines uniquely the field of matter as well as the electromagnetic field. In the singular case the decomposition is not unique: one of the tensors (each of which represents in a sense radiation) can be chosen to a certain degree arbitrarily.

4. Professor L. L. Dines: *A mininum problem from physiology.*

The sensitive portion of the retina consists of a collection of circular surfaces ("cone" surfaces) of equal known radii, uniformly distributed, the intervening area being non-sensitive. Assuming known the size of the smallest circular image which can be seen, it is desired to know what is the minimum area of cone surface which such an image can occupy. This question, proposed by a colleague of the author, is considered in the present note.

5. Professor E. L. Dodd: *Probability as expressed by asymptotic limits of pencils of sequences.*

In order to deal with probability without recourse to the much criticized notion of "cases equally likely," a set of sequences is introduced, called a *pencil of sequences*, inasmuch as they radiate (figuratively speaking) from a common origin. If H means "heads," and T means "tails," an individual sequence, $HHTH \cdots$ is not conceived as giving rise to any limit. But for a pencil an actual limit is involved when only an evanescent percentage of the sequences fail to conform to certain inequalities. An exact and rather simple formulation—avoiding all non-mathematical phrases and conceptions, indeed, *avoiding the word "probability" itself*—is given to the Bernoulli theorem, its extensions, and its inverse. This involves only the counting of sequences, and the use of ordinary limits. This may give the student of probability a clearer conception of the Bernoulli inverse than he gets from the following statement: With arbitrary *a priori* probabilities, there is ultimately a very high *a posteriori* probability that the *empirical* probability will closely approximate that *unknown* probability that gives rise to the *productive* probability. References are made to the work of Cantelli, von Mises, Borel, Lomnicki, and Steinhaus.

6. Professor W. E. Milne: *On the degree of convergence of expansions on an infinite interval.*

The author studies the rate of convergence of expansions in terms of the characteristic functions which are associated with the differential equation $d^2u/dx^2+(\lambda-q(x))u=0$ and which are also subject to the conditions $\lim_{x=\pm\infty}u=\lim_{x=\pm\infty}u'=0$. With appropriate restrictions of the function $q(x)$ an approximate formula is obtained relating the nth characteristic number λ_n with n. In particular it is found that the order of magnitude of λ_n is always less than n^2. It is shown that if $f(x)$ has a continuous kth derivative of bounded variation in the infinite interval, and satisfies certain conditions at infinity, then the remainder after n terms of the expansion of $f(x)$ is $O(\lambda^{-k/2})$ in any fixed finite interval.

7. Professor H. T. Davis: *Properties of the Laurent expansion of differential operators of infinite order with constant coefficients.*

This paper exhibits some simple properties of the development of the symbolic operator $G(z)=1/F(z)$, in which we abbreviate $F(z)=a_0+a_1z+a_2z^2+\cdots$, where $z=d/dx$, in annuli formed by concentric circles about the origin. The following theorem is proved: If $H(z)$ designates a Laurent expansion of $G(z)$ in an annulus formed by two concentric circles about the origin and if $G(z)$ is any other expansion about the origin, then the function $U(x)=\{H(z)-G(z)\}\rightarrow f(x)$, where $f(x)$ is arbitrary to within the limits of the existence of the right hand member, is a solution of the homogeneous equation, $F(z)\rightarrow u(x)=0$. The case where $G(z)$ is a rational function is shown to yield the fundamental operator of the Heaviside operational calculus when the expansion is that in the region exterior to all the poles of the function.

8. Dr. W. D. Baten: *The evaluation of certain definite integrals by the use of probability functions.*

The object of this paper is to present three methods of evaluating certain definite integrals by using probability functions. The first method consists in finding the probability law or function for the sum of n independent variables, which are each subject to given probability laws, by Mayr's method and comparing this with the probability for the same sum obtained by Dodd's method. Method two consists in finding the probability law for n independent variables, which is expressed as an integral, and then allowing n to be equal to 1. Method three is founded on setting up a probability function containing the integral to be evaluated as a factor, determining a second probability function defined over the same interval, and then finding a third probability function for a chosen relation of the variables under consideration. The integral need not be a factor of the first function, but should arise somewhere in the process. The first two methods, as far as the writer knows, seem to be new.

9. Professor D. R. Davis: *Integrals whose extremals are a $2n$-parameter family of curves.*

One considers here a given $2n$-parameter family of arcs of the form

$y_i = y_i(x, a_1, \cdots, a_n, b_1, \cdots, b_n)$ $(i = 1, \cdots, n)$, and seeks the properties which are sufficient to ensure that they are the extremal arcs of an integral of the form $I = \int_{x_1}^{x_2} f(x, y_1, \cdots, y_n, y_1', \cdots, y_n')dx$. These properties are determined under the hypothesis that the differential equations $y_i'' = F_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n')$, found by eliminating the $2n$ parameters $a_1, \cdots, a_n, b_1, \cdots, b_n$, are the solutions for y_i'' of a system of equations of the form $H_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n', y_1'', \cdots, y_n'') = 0$, whose equations of variation are self-adjoint. The general type of integral whose extremals are the arcs $y_i = a_i x + b_i$ $(i = 1, \cdots, n)$ is completely determined.

10. Professor D. R. Davis: *Determining integrals whose extremals are parabolas.*

In this paper the inverse problem for the trajectory of a particle under the action of gravity is considered. Properties of its characteristic differential equations are developed and a general type of solution is obtained. From the latter the integrals desired are completely determined.

11. Professor W. L. Hart: *Infinite systems of non-linear equations and related integral equations.*

Let x and y represent points in real Hilbert space of infinitely many dimensions. Without employing convergent infinite determinants, the author establishes the unique existence of a solution $x(y)$ of the following infinite system of equations (1): $f_i(x, y) = 0$ $[x \equiv (x_1, x_2, \cdots); y \equiv (y_1, y_2, \cdots); i_1 1, 2, \cdots]$ in a neighborhood of a solution $(x = a, y = b)$. The most important of the hypotheses used are that the matrix $F \equiv (\partial f(x, y)/\partial x)$ is limited in the Hilbert sense, that $F(a, b)$ possesses a unique limited reciprocal, and that $[F(x, y) - F(a, b)]$ possesses a bound $g(x, y)$, continuous at (a, b), where $g(a, b) = 0$. By use of the properties of limited multilinear forms, it is demonstrated that the results for (1) can be applied in the case of various infinite systems of non-linear equations in infinitely many unknowns. Each non-linear system considered can be used to establish the existence of continuous solutions of a certain non-linear integral equation. The transition from an integral equation to a system of the form (1) is made by the usual method which introduces the Fourier constants of the continuous functions involved relative to some complete orthogonal system.

12. Mr. C. M. Erikson: *A direct solution of systems of homogeneous linear difference equations having constant coefficients.*

The paper gives in detail the method and theory of obtaining the solutions of simultaneous systems of linear difference equations with constant coefficients. The first case treated is the simple one in which the characteristic equation, $D(\lambda) = 0$, has distinct roots. The solutions are expressed in terms of first minors of the characteristic determinant $D(\lambda_i)$. The next case considered is the one in which $D(\lambda) = 0$ has a triple root. The most general case is then treated, under hypotheses that include every

special case that could arise. In this case the solutions are obtained corresponding to a multiple root $\lambda=\lambda_i$ of the characteristic equation $D(\lambda)=0$, where entirely general assumptions are made regarding the vanishing of minors of different orders in the characteristic determinant $D(\lambda_i)$. The solutions thus obtained are shown to form a fundamental set. The problem is treated without making use of the theory of transformations.

13. Professor C. C. Camp: *On multiparameter expansions associated with a differential system and auxiliary conditions at several points in each variable.*

For $p=2$ the system $dX_i/dx_i+\sum_{j=1}^{p}\lambda_j a_{ij}(x_i)X_i=0$, $(i=1, 2, \cdots, p)$, and boundary conditions of the Wilder type in each lead to suitable characteristic solutions provided the a_{ij} are properly restricted. It is sufficient if $a_{ij}(x_i)$ is a real, integrable function, either identically zero or of constant sign, which maintains its mean value in each segment of the boundary interval $a_i \leqq x_i \leqq b_i$, and the determinant $\left|\int_{a_i}^{b_i} a_{ij}dx_i\right| \neq 0$. If the k_i segments have lengths commensurable with b_i-a_i and are the same for $i=1, 2$, the factors containing one parameter may be eliminated by Sylvester's method, so that the characteristic values of the other will be zeros of an entire function. For $k_i>3$ the restrictions on $a_{ij}(x_i)$ may be lightened in the interior subintervals.

14. Dr. Lincoln LaPaz: *Problems of the calculus of variations with prescribed transversality conditions.*

The transversality condition for a problem of minimizing the integral

$$I=\int_{x_1}^{x_2} f(x, y_1, \cdots, y_n, y'_{1n}, \cdots, y'_n)dx \tag{1}$$

is

$$(f-y'_\mu \partial f/\partial y'_\mu)dx+\partial f/\partial y'_\mu dy_\mu=0 \tag{2}$$

(μ is umbral, ranging from 1 to n). In this paper it is shown that a relation

$$dx+T_\mu\,(x, y_1, \cdots, y_n, y'_1, \cdots, y'_n)\,dy'_\mu=0, \quad 1+y'_\mu T_\mu \neq 0, \tag{3}$$

can be the transversality relation (2) of a problem (1) only in case the T_i are such that a certain system of n linear homogeneous partial differential equations is a complete system. Necessary and sufficient conditions for this system to be complete are that

$$y'_\mu(T_m \partial T_k/\partial y'_\mu - T_k \partial T_m/\partial y'_\mu)+(\partial T_k/\partial y'_m-\partial T_m/\partial y'_k)=0, \tag{4}$$

$(m, k=1, \cdots, n, m<k)$. For n functions T_i satisfying (4) the most general integral (1) for which (3) is the transversality condition is determined. The results obtained supplement those found by a different method by Rawles (Transactions of this Society, vol. 30 (1928), pp. 765–784).

15. Dr. Lincoln LaPaz: *An inverse problem of the calculus of variations for multiple integrals.*

The Lagrange equation of a problem of minimizing an n-tuple integral $J=\int_{(n)} f(x_1, \cdots, x_n, z, p_1, \cdots, p_n)dx_1 \cdots dx_n$, $(p_i=\partial z/\partial x_i)$, must be of the

form $A_{\mu\nu}(x_1, \cdots, x_n, z, p_1, \cdots, p_n)\, p_{\mu\nu}+B(x_1, \cdots, x_n, z, p_1, \cdots, p_n)=0$, ($p_{ij}=\partial^2 z/\partial x_i \partial x_j$, $A_{ij}=A_{ji}$, $i, j, \mu, \nu=1, \cdots, n$, μ and ν umbral), with an equation of variation which is self-adjoint on every hypersurface $z=z(x_1, \cdots, x_n)$. It is shown that if an equation $F(x_1, \cdots, x_n, z, p_1, \cdots, p_n, p_{11}, \cdots, p_{nn})=0$ has the above properties, there is always an integral J having $F=0$ as its Lagrange equation. The most general such integral has an integrand function of the form $f=C(x_1, \cdots, x_n, z, p_1, \cdots, p_n)+D(x_1, \cdots, x_n, z)+E_\mu(x_1, \cdots, x_n, z)p_\mu+\partial w_\mu/\partial x_\mu$, where C is a particular solution of the system $\partial^2 C/\partial p_i \partial p_j=A_{ij}$, and (D, E_i) is a particular solution of $\partial D/\partial z-\partial E_\mu/\partial x_\mu=\partial^2 C/\partial p_\mu \partial x_\mu+(\partial^2 C/\partial p_\mu \partial z)p_\mu-\partial C/\partial z-B$, and where the w_i occurring in the sum of total derivatives with respect to x_i are arbitrary functions of $x_1, \cdots, x_n, z$ alone. The method of proof is similar to that used by Davis (Transactions of this Society, vol. 30 (1928), pp. 710–736) in connection with an analogous question for simple integrals $I=\int_{x_1}^{x_2} f(x, y, z, y', z')dx$.

16. Mr. H. P. Thielman: *The application of fractional operations to a class of integral equations occurring in physics.*

This paper is concerned with the theory of a class of Volterra integral equations of the first kind occurring in physics. The methods used are those of fractional operators and Volterra's calculus of composition. The theorems and results of Wrinch and Nicholson (Philosophical Magazine, (7), vol. 4(1927), pp. 531–560) are obtained as special cases of the author's more general and elegant theory.

17. Mr. L. B. Robinson: *On a lacuna in the theory of systems of partial differential equations.*

In his book *Les Systèmes d'Equations aux Dérivées Partielles*, Riquier examines in detail the passivity conditions of such systems, but does not consider the case where the passivity conditions entrain relations among the independent variables. The author has partly succeeded in filling in this lacuna. Consider the three equations

(1) $\partial u/\partial x=f(x, y, z, u)$, $\partial u/\partial y=\phi(x, y, z, u)$, $\partial u/\partial z=\psi(x, y, z, u)$.

This case illustrates that for n equations. Supposing certain relations between the independent variables

$$(2)\qquad \begin{aligned} &\partial u/\partial x=f(x,\ \eta(x),\ \zeta(x);\ u),\\ &\partial u/\partial y=\phi(\eta_{-1}(y),\ y,\ \zeta[\eta_{-1}(y)];\ u),\\ &\partial u/\partial z=\psi(\zeta_{-1}(z),\ \eta[\zeta_{-1}(z)],\ z;\ u), \end{aligned}$$

where η and η_{-1} and ζ and ζ_{-1} are pairs of inverse functions such that

$$(A)\qquad \begin{aligned} &\eta_{-1}(y)dy=x\eta'(x)dx,\ \zeta_{-1}(z)dz=x\,\zeta'(x)dx,\\ &\eta[\zeta_{-1}(z)]dz=\eta(x)\zeta'(x)dx,\quad \zeta[\eta_{-1}(y)]dy=\zeta(x)\eta'(x)dx, \end{aligned}$$

u can be computed in terms of x and definite integrals involving $\eta(x)$ and $\zeta(x)$, or in terms of y and definite integrals involving $\eta_{-1}(y)$ and $\zeta[\eta_{-1}(y)]$, or in terms of z and definite integrals involving $\zeta_{-1}(z)$ and $\eta[\zeta_{-1}(z)]$, the definite integrals arising in the process of solving system (2) by successive approximations. Now keeping in mind the equalities (A), we find $\eta(x)$

and $\zeta(x)$ by forming two integro-differential equations which can easily be solved, as the kernals do not depend on the upper limits of integration. This method can be extended to systems of equations of the second order.

18. Professor George Rutledge: *The basic power series of interpolation.*

The coefficients of the polynomial $P^{(2n)}(x)$, determined by the values $f(-n), \cdots, f(0), \cdots, f(n)$, are expressed in the form of series the terms of which are independent of n. Conditions for convergence of $P^{(2n)}(x)$ are readily derived from this form of expansion of the coefficients. This paper is a direct continuation of the paper of similar title published in the Transactions of this Society (vol. 26 (1924), p. 113).

19. Dr. Hillel Poritsky (National Research Fellow): *On certain polynomial approximations to analytic functions of a complex variable.*

This paper deals with the convergence to $f(z)$ of certain polynomials $P_{m,n}(z)$ as n becomes infinite, where $P_{m,n}(z)$ is of degree $mn-1$ at most and is uniquely determined by mn conditions requiring that its derivatives of order $0, m, \cdots, m(n-1)$ at m given fixed points $a_1, a_2, \cdots, a_m$, no two of which are alike, be equal respectively to the same order derivatives of $f(z)$ at the same points. The case $m=1$ leads essentially to Taylor's expansion of $f(z)$ about $z=a_1$, but is not typical of the general case $m>1$. Thus, mere analyticity of $f(z)$ in any region will not suffice to ensure the convergence of $P_{2,n}(z)$ to $f(z)$ as n becomes infinite. This may be seen by putting $a_1=0$, $a_2=\pi$, and choosing $f(z)=\sin z$; $P_{2,n}(z)$ will then identically vanish. Sufficient conditions for convergence of $P_{m,n}(z)$ to $f(z)$ as n becomes infinite, it is shown, are that $f(z)$, in addition to being an integral function of z, satisfy an inequality $|f(z)|<Ce^{k|z|}, k<K$, where C, k are constants, and K is a constant determined by $a_1, a_2, \cdots, a_m$; the convergence then takes place for all z. For $m=2$, $K=\pi/|a_1-a_2|$. In the case $m=2$ necessary conditions that are nearly sufficient are also found.

20. Dr. Morris Marden (National Research Fellow): *On the zeros of solutions of a generalized Laplacian differential equation.*

This article aims to furnish specific means for describing qualitatively the distribution of the zeros of any solution of the equation

$$d^2U/dz^2+(a+b/z)(dU/dz)+(c+d/z)+f/z^2)U=0, \tag{1}$$

where z is a complex variable, but a, b, c, d, f are real constants. These means are constructed as follows: Through the transformation

$$U=z^{-b/2}e^{-az/2}w,$$

equation (1) is converted into one of the form

$$d^2w/dz^2+(A+B/z+C/z^2)w=0. \tag{2}$$

Every solution of the new equation has, with the possible exception of the

point $z=0$, the same zeros as the corresponding solution of (2). Now by use of Hille's method of "zero-free regions" (Arkiv för Matematik, vol. 18 (1924), pp. 1–56) information can be gathered which, with a few moments' additional work, enables one to locate regions where neither any particular solution of (2) nor its derivative vanishes. The efficacy of the so-constructed means is illustrated by the study of solutions of (2) with the property $W(0)=0$ or $W(+\infty)=0$, where $W(z)=\overline{w(z)}\,w'(z)$.

21. Professor R. E. Langer: *A class of irregular boundary problems of the second order.*

The characteristic parameter values for the boundary problem of an ordinary linear differential equation are determined as the roots of a relation (1): $\sum[\alpha_i]e^{\rho\beta_i}=0$, in which ρ is the parameter, the β_i are complex constants, and $[\alpha_i]=\alpha_i+\alpha_{i1}/\rho+\cdots$. With this transcendental equation there is associated in the complex plane the convex polygon which has vertices only at points of the set β_i and which contains all the points β_i in its interior or on its perimeter. The boundary problem is called "regular" if in equation (1) $\alpha_i\neq 0$ for those values of i for which β_i is a vertex of the polygon. Only for particular differential equations has the irregular boundary problem been studied. The present paper considers the problem of the general differential equation of the second order for the irregular cases in which the quantities $[\alpha_i]$ vanish to various degrees in ρ but not identically. It is shown that the associated expansions of arbitrary functions are in these cases summable by suitable means under hypotheses similar to those which ensure convergence of the expansions in the regular cases.

22. Professor L. E. Ward: *On the differentiation and integration of certain formal series.*

The series considered are those associated with the differential system $u'''+\rho^3u=0$, $u(0)=u'(0)=u(\pi)=0$. Necessary and sufficient conditions that the third derived series of the formal series for a function capable of being expanded in a series of the characteristic functions of this differential system should be the formal series for the third derivative of the function are obtained. It is proved that the threefold term-by-term indefinite integration of the formal series for a function yields the formal series for a certain function, and a formula for the latter is found.

23. Professor L. E. Ward: *On the determination of the formal coefficients in a certain expansion problem.*

If the functions $u_k(x)$ are the characteristic functions of the differential system $u'''+\rho^3u=0$, $u(0)=u'(0)=u(\pi)=0$, and $f(x)$ is a function capable of expansion in a series of these characteristic functions, and if $R_n(x)=f(x)-\sum_{k=1}^{n}a_ku_k(x)$, it is shown that $\int_0^\pi R_n(x)R_n(\pi-x)dx$ has an extremum when the a_k's are the coefficients of the formal series for $f(x)$.

24. Professor J. V. Uspensky: *On a new method for finding units in cubic rings of a negative discriminant.*

The basic idea of this new method can be best explained for the simplest

case of a cubic ring $[1, \theta, \theta^2]$ where θ denotes a real root of the equation $\theta^3=a$. Set $\phi=x+y\theta+z\theta^2$, $\phi=\phi'\phi''$, ϕ' and ϕ'' being conjugate forms to ϕ, and similarly $\omega=Z+Y\theta+Z\theta^2, \Omega=\omega'\omega''$. Following Hermite's suggestion we introduce the ternary quadratic form $f=2\phi+\phi^2/\Delta$ containing a variable positive parameter Δ, whose adjoint form differs from $F=\omega^2+2\Omega/\Delta$ by an irrelevant numerical factor. But instead of considering successive minima of the form f for a continuously increasing parameter, as Hermite suggests, it seems more advantageous to consider systems of integers x, y, z; X, Y, Z satisfying for every Δ the conditions $f \leqq 4(a^2/\Delta)^{1/3}$, $F \leqq 4(a^2/\Delta^2)^{1/3}$, $Xx+Yy+Zz=0$. It can be shown that, starting with $x=1$, $y=0$, $z=0$; $X=0$, $Y=0$, $Z=1$ for $\Delta=1$ we can reach systems x, y, z; X, Y, Z such that either $\phi=x+y\theta+z\theta^2$ or $\omega=Z+Y\theta+X\theta^2$ will be a fundamental unit. When applied to numerical examples the new method seems to be quite satisfactory and not inferior to any known method that can be used for the same purpose.

25. Professor H. S. Vandiver: *On power characters of singular integers in a properly irregular cyclotomic field.*

A properly irregular cyclotomic field is a field defined by $\zeta=e^{2\pi i/l}$, l an odd prime, which is irregular and such that certain units E_n, $n=1, 2, \cdots, (l-3)/2$ (defined as by the writer, Proceedings of the National Academy of Sciences, vol. 15 (1929), p. 202) are not lth powers of units in $k(\zeta)$. In the present paper the author shows how to determine in algebraic form the lth power characters of any singular integer in a properly irregular cyclotomic field $k(\zeta)$, with respect to any ideal in this field prime to l. Heretofore this determination has only been made for the cases where the ideals belonged to an exponent prime to l. The paper will appear in the Transactions of this Society.

26. Professor H. S. Vandiver: *On the composition of the class-group of ideals in an irregular cyclotomic field.*

If the class number h of an irregular cyclotomic field is expressed in the form $l^s g$, where g is prime to l and the cylotomic field is defined by $e^{2\pi i/l}$, with l an odd prime, then the gth powers of all ideals in the field may be separated into classes which form an abelian group under multiplication. In the present paper the invariants of this group of classes are shown to depend on the divisibility of certain Bernoulli numbers by powers of l, in the case where the cyclotomic field is properly irregular. Properties of a basis for this group are also given. The investigation gives a number of other results concerning the class-field of a cyclotomic field and a basis for the singular primary numbers in the field. The paper will appear in the Transactions of this Society.

27. Professor H. S. Vandiver: *Some properties of a certain system of independent units in a cyclotomic field.*

Pollaczek (Mathematische Zeitschrift, vol. 21 (1924), pp. 1–38) proved

that in the field $k(\zeta),\zeta=e^{2\pi i/l}$, l an odd prime, there exists a system of fundamental real units with the following property:

$$\eta_i^{s-s^{2i}}=(\eta')_i^l, \qquad (i=1, 2, \cdots, (l-3)/2),$$

using the Kronecker-Hilbert symbolic powers. In the present note it is shown that certain units γ_i, $i=1, 2, \cdots, (l-3)/2$, which are analogous to those defined by Hilbert (*Bericht*, p. 432), may be expressed in forms involving the η's. This paper will appear in the Annals of Mathematics.

28. Professor H. S. Vandiver: *Some theorems concerning properly irregular cyclotomic fields.*

This article contains a number of theorems concerning properly irregular cyclotomic fields. Statements of these theorems with proofs indicated appeared in Proceedings of the National Academy of Sciences, vol. 15 (1929), p. 202–207. Full proofs will appear later in the Transactions of this Society.

29. Professor H. S. Vandiver: *Summary of results and proofs on Fermat's last theorem.* Fourth Paper.

This note contains a number of results concerning Fermat's last theorem. Statements of the theorems without proofs appeared in Proceedings of the National Academy of Sciences, vol. 15 (1929), pp. 108–109. The proofs of the results will appear in the Transactions of this Society.

30. Professor Lennie P. Copeland: *Matrix conditions that the ternary cubic locus may have multiple points.*

It is the purpose of the present paper first to determine a minimum set of necessary and sufficient conditions that the general ternary cubic locus (degenerate) may have exactly two double points, and second to show that there is a locus analogous to the ternary Hessian in the case of the degenerate polar cubic, since the locus of points whose polar cubic with respect to a given quartic break up into three straight lines is expressible as the three necessary and sufficient conditions that a cubic be factorable into linear factors.

31. Professor C. G. Latimer: *On the class number of the cubic cyclotomic field.*

In a former paper, the writer determined the prime ideals in the cubic Galois, or cyclotomic, field. Employing this result, the author determines the class number of the field in terms of the regulator. The method is similar to the well known method of treating the same problem for the quadratic field.

32. Professor G. W. Smith: *Some results obtained by generalizing the analytical definition of a conic.*

The ordinary analytical definition of a conic is generalized by considering the eccentricity as a function of either or both of the coordinates of the generating point. Some general properties of the loci whose equations are

thus determined are given. Certain special functions for the eccentricity lead to interesting results in both rectangular and polar coordinates. It is shown that any curve whose equation is $f(x, y)=0$ may be considered as having a directrix and a focus, the eccentricity being variable.

33. Mr. R. G. Smith and Professor E. B. Stouffer: *A proof of a theorem of Halphen and its extension to n dimensions.*

If two curves in ordinary space have contact of order m at a point, there exists a unique plane from any point of which one can project the two curves by means of two cones with contact of order $m+1$. This theorem was proved by Halphen in 1880 (Journal de l'Ecole Polytechnique, vol. 47, p. 25). In the present paper a simpler proof of the same theorem is given, and the theorem is also extended to curves in contact in space of n dimensions.

34. Dr. Marie J. Weiss (National Research Fellow): *The limit of transitivity of a substitution group.*

In this paper inequalities giving the relation between n, the degree of a t-ply transitive group, and t, the transitivity of the group, are found. The following theorems are proved: (1) Let the order of the subgroup G_t that fixes t letters of a t-ply transitive group of degree n be divisible by an odd prime. Then if $t\geqq 8$, $(n-2t+1)/2\geqq t!/\{\delta!(t-\delta)!\}$, where δ is the greatest integer less than the quantity $t-(t-k+1)(\log 2)/(k+\log 2)$, k an integer such that $5\leqq k\leqq t-1$; or $(n-2t+1)/2\geqq t2^{\alpha}$, α an integer $\geqq 4$ and $\geqq k-3-\log k/\log 2$; or $(n-2t+1)/2\geqq 2^{\gamma}$, γ an integer $\geqq 4$ and $\geqq t-3-\log t/\log 2$. (2) Let G_t be of order 2^{μ}. Then if $t\geqq 8$, $(n-t)/2\geqq t!/\{\beta!(t-\beta)!\}$, where β is an integer chosen such that $5\leqq\beta\leqq t-3$, or chosen to be one unit less than the least value of β which satisfies the inequality

$$[(\beta+1)/2]!\geqq 2t!/\{\beta!(t-\beta)!\};$$

$$\text{or } n-t\geqq[(t+1)/2]!;\ \text{or } n-t\geqq t!/[t/2]!^2.$$

The symbol $[s]$ denotes the integral value of s.

35. Dr. L. W. Griffiths: *A theorem on extended polygonal numbers.*

If m is a positive integer, and $x=1, 0, -1, -2, -3, \cdots$, the integers $e(x)=x+m(x^2-x)/2$ are all the extended polygonal numbers of order $m+2$ and the first polygonal number, $e(1)=1$. Dickson has proved that every positive integer is the sum of m values of $e(x)$ if $m\geqq 6$, but of $m+1$ values if $m=3, 4, 5$. A generalization of this theorem is suggested by the following restatement. Let $n, a_1, \cdots, a_n$ be positive integers, and $e_1, \cdots, e_n$ be values of $e(x)$; then the function $f=a_1e_1+\cdots+a_ne_n$ is universal when $n=m\geqq 6$, or $n=m+1$ if $m=3, 4, 5$, and $a_1=\cdots=a_n=1$. In $+a_n\leqq m$, and also for $m=3, 4, 5$, and $a_1+\cdots+a_n\leqq m+1$. For $m=1, 2$ the problem reduces to that of polygonal numbers.

36. Professor L. S. Hill: *Concerning linear transformations whose coefficients are matrices.*

The term "ring" being used as specified in Hasse, *Höhere Algebra*, part 1, pp. 7–9, any finite or infinite ring containing a unit element (which is necessarily unique) is here called a "scale." In particular, every field, and every "Integritätsbereich," is a scale. The set of all square matrices, of any assigned order, in which the elements are elements of the scale S is called a "range," Γ. The ordinary type of matrix algebra is set up in Γ, and attention is directed to the transformation

$$T \qquad y_1 = a_{11}x_1 + \cdots + a_{1n}x + a_1, \cdots, \; y_n = a_{n1}x + \cdots + a_{nn}x_n + a_n,$$

defined by n linear equations in which the x_i, y_i, a_{ij}, a_i denote matrices of Γ. Conditions for the existence of a unique inverse transformation T^{-1}, and means to determine its equations, are given. A facile construction is presented for transformations T, of any desired complexity, which are involutory (that is, of period 2). All results are valid in ranges of matrices whose elements are in turn matrices, the elements of these elements being perhaps again matrices, etc. An application is made in the preparation of cipher transformations offering extraordinary resistance to cryptanalysis.

37. Professor A. D. Campbell: *Note on linear transformations of n-ics in m variables.*

This paper has appeared in the September-October number of this Bulletin.

38. Mr. D. H. Lehmer: *Numerical integration as applied to Dirichlet series.*

In the literature on the analytic theory of numbers, there are a large number of "identities" between certain Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ and simple expressions involving Riemann's ζ function. The methods used in proving these identities vary considerably; for the most part, however, use is made of the infinite product definition of $\zeta(s)$. The purpose of this note is to indicate a principle of unification by means of which all these identities fall into classes such that all identities of the same class may be obtained simultaneously. This unifying principle is based on Bougaieff's concept of numerical integration of a function $f(n)$ defined by the relation $F(n) = \sum_{\delta/n} f(\delta)$. We consider a sequence $f_r(n)$ defined for all positive and negative integer values of r by the relation $f_{r+1}(n) = \sum_{\delta/n} f_r(\delta)$, the fundamental function $f_0(n)$ being specified. In this way the function $f_0(n)$ defines a class of Dirichlet series $\sum_{n=1}^{\infty} f_r(n) n^{-s}$. Most of the known identities belong to three classes defined by $f_0(n) = n^k$, $\log n$, $\lambda(n)$, the latter being Liouville's function. By introducing new functions f_0 one obtains the rest of the known identities as well as many new ones.

39. Professor Raymond Garver: *Quartic equations with the alternating group.*

Baker and Seidelmann have given expressions for the most general quartic equation whose Galois group is the alternating group. The present paper shows that Baker's form actually does not represent all such quartics, while Seidelmanns' form does. The verification of this last fact

is entirely independent of Seidelmann's work, and requires only rather elementary algebraic manipulations.

40. Professor Raymond Garver: *Transformations on cubic equations.*

This paper, which will be published in the American Mathematical Monthly, studies exceptional Tschirnhaus transformations on cubic equations which are not equivalent to linear fractional transformations. A number of statements in the literature deny the existence of such transformations, and they are perhaps worth mentioning on that account as well as because they lead to certain rather interesting results.

41. Professor W. E. Roth: *On the unilateral equation in matrices.*

Let A_i, $i=0, 1, \cdots, p$, be rectangular matrices of m rows and n columns and let X be an unknown square matrix of order n; then the equation $\sum_{i=0}^{p} A_i X^{p-i}=0$ is said to be unilateral (Sylvester, Philosophical Magazine, vol. 17 (1884), p. 392). The present paper develops the necessary conditions for the existence of X satisfying this equation and in particular extends a theorem due to Sylvester regarding the latent (characteristic) roots of the unknown matrix (Comptes Rendus, vol. 99 (1884), pp. 621–631). This theorem for our more general equation may be stated thus: If $\sum_{i=0}^{p} A_i \lambda^{p-i}$ is the matrix obtained from the left member of the equation above when X is replaced by the scalar variable λ, then the latent roots of X must be values of λ for which the rank of $\sum_{i=0}^{p} A_i \lambda^{p-i}$ is less than n. The converse is not true. The results apply equally well when the unknown matrix of order m occurs on the left of all terms of the given equation.

42. Mr. John McDonnell: *Note on Fermat's last theorem.*

The author proves that if $x^p+y^p+z^p=0$, then (1) if $yz+zx+xy \not\equiv 0 \pmod{p}$, and r be any factor of x^2-y^2, then $r^{p-1} \equiv 1 \pmod{p^2}$, and (2) if $x(y-z)(x^2+y^2) \not\equiv 0 \pmod{p}$, and r be any factor of x^2+yz, then $r^{p-1} \equiv 1 \pmod{p^2}$. The proof resembles a proof of Furtwängler's for a similar result.

43. Professor H. A. Simmons: *Maximum numbers associated with a certain diophantine equation.*

This paper presents for the equation

$$\sum(1/(x_1 x_2 \cdots x_{n-1}))=b/[(m+1)b-1]$$

a solution in positive integers which has three special properties, namely: it contains the largest number x that can appear in any solution in which $x_1, x_2, \cdots, x_{n-1}$ are positive integers, with x_n only restricted to be a real number $\geqq x_i$, $i=1, 2, \cdots, n-1$; and, with the same hypotheses on the x's, the solution found maximizes the sum $x_1+x_2+\cdots+x_n$ and the product $x_1 x_2 \cdots x_n$. The final section of the paper exhibits for the equation $\sum(1/(x_1 x_2 \cdots x_r))=b/[(m+1)b-1]$, r being less than n, the

number of variables in the symmetric function, a solution in positive integers which includes as a special case the solution whose properties were mentioned above.

44. Mr. J. H. Roberts: *A non-dense continuum.*

The author has shown that if M is a square plus its interior then there exists an upper semi-continuous collection G of mutually exclusive continua such that (1) every point of M belongs to some continuum of G, (2) G is topologically equivalent to M, and (3) each continuum of G contains more than one point. This question arises: "Is it necessary that a given continuum M contain a domain in order that such a set G exist?" In the present paper an example is given which shows that the answer to the above question is in the negative.

45. Dr. Leo Zippin: *Continuous curves and the Jordan curve theorem.*

It is shown that if a continuous curve C, is such that (1) every simple closed curve of C disconnects C, and (2) no arc of C disconnects C, then C satisfies the Jordan Curve-Theorem and is homeomorphic with the complement on a simple closed surface of a closed and totally disconnected point set (possibly vacuous, if C is compact); these surfaces the author has previously termed cylinder-trees. It is proved that a necessary and sufficient condition that a continuous curve C containing at least one simple closed curve be a cylinder-tree, is that if K is any simple closed curve of C, then $K \subset \overline{C-K}$ and $C-K$ is the sum of two components. The author shows that the following postulate-set characterizes the number-plane: Axioms 1–5 of R. L. Moore's $\Sigma-1$, and Axiom 8′: if K is a simple closed curve of the space S, then $K \subset \overline{S-K}$, and $S-K$ is the sum of two components at least one of which is compact. Further, if in this system Axiom 2 be replaced by Axiom η': a region is a connected set of points not a single point, Axiom 3 may be replaced by 3′: if x is a point of S, $S-x$ is connected, or else by 3″: if m is an arc of S, there is a point x of m such that $S-x$ is connected.

46. Professor G.T.Whyburn: *Concerning continuous curves and denodular sets.*

A connected and locally connected set is denodular if all its nodular elements (that is, elements corresponding to the cyclic elements of a continuous curve) are points. It has previously been shown by the author that a denodular set is highly similar to a connected subset of an acyclic continuous curve. In this paper a study is made of the conditions under which a continuous curve may be expressed as the sum of a denodular set and another set H of simple structure. For example, H may be finite or may be null-dimensional. In this way characterizations of various types of continuous curves are obtained. For example, it is shown that a continuous curve is one-dimensional if and only if it is the sum of a denodular set and a null-dimensional set.

47. Professor R. L. Wilder: *Extension of a theorem of Mazurkiewicz.*

Mazurkiewicz has recently shown (Fundamenta Mathematicae, vol. 13, pp. 146–150), in answer to a question proposed by Knaster, that if A is a closed set in E_n which is homeomorphic with a subset of E_{n-1}, then all points of A are accessible from $E_n - A$. It is shown in the present paper that the points of A are regularly accessible from $E_n - A$. In the proof the following fundamental lemmas are established: (1) If a is a point of A, and S is a spherical neighborhood of a, then if the component C of $S \cdot A$ which contains a separates S, the set C is homeomorphic with a connected open subset of E_{n-1}. (2) If A separates two points P and Q in some spherical neighborhood S, there exists a component of $A \cdot S$ which separates P and Q in S.

48. Professor R. L. Wilder: *Concerning perfect continuous curves.*

The following results hold in any locally compact metric space: (1) Every compact connected subset of a perfect continuous curve has property S. (2) Every component of a quasi-open subset of a perfect continuous curve is arcwise connected im kleinen. (3) In order that a continuum M should be a perfect continuous curve, it is necessary and sufficient that the boundary points of the components of quasi-open subsets of M should be regularly accessible. (4) If K is a closed and bounded set lying in a component Q of a quasi-open subset of a perfect continuous curve, then K is a subset of a continuous curve which itself lies in Q.

49. Professor R. L. Wilder: *A converse of the Jordan-Brouwer separation theorem in three dimensions.*

This paper presents a converse, in E_3, of the Jordan-Brouwer separation theorem. (See J. R. Kline, this Bulletin, vol. 34, pp. 156–159.) The result may be stated as follows: Let K be a closed and bounded set in E_3 such that (1) the Betti numbers (mod 2) $R^0(E_3 - K)$ and $R^1(E_3 - K)$ are respectively equal to 2 and 1, (2) if D is a component of $E_3 - K$, the 0-chains in D are uniformly homologous to zero in D*, and every point of K is a limit point of D. Then K is a simple closed surface. Another statement of the result, not limiting the number of complementary domains, but strengthening the accessibility conditions, is also given.

50. Dr. W. L. Ayres (National Research Fellow): *On the density of the cut points and end points of a continuous curve.*

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* That is, for any number $e > 0$, there exists a number d such that if L^0 is a 0-chain of D of diameter less than d, L^0 bounds a chain L^1 in D of diameter less than e. This property is equivalent to uniform connectedness im kleinen here, and does not imply that a set is connected.

The author proves the following relations between the density of the end points and that of the cut points of a continuous curve M: the set K of all cut points of M is dense in M if, and only if, the end points of M are dense in $M-K$. In order that the end points of M be dense in M it is necessary and sufficient that the set of all cut points of Menger-Urysohn order ≥ 3 be dense in M. When the end points are dense in M, the set of all cut points of order 2 is zero-dimensional. And conversely, when the set of cut points of order 2 is zero-dimensional and the set of all cut points is dense in M, then the end points are dense in M.

51. Dr. W. L. Ayres: *On joining subsets of a continuous curve by arcs and simple closed curves.*

It is shown that if K is a finite subset of m points of a continuous curve M and no two points of K may be separated in M by a set of $m-1$ points of M, then there is a simple closed curve of M containing K. If no two points of K are separated by $m-2$ points, there is an arc of M containing K. Examples are given to show that the numbers $m-1$ and $m-2$ cannot be reduced, but for the special case of a plane continuous curve M the following sharper theorem is true: If M remains connected on the omission of any pair of points, then every finite subset lies on a simple closed curve of M.

52. Dr. W. L. Ayres: *On continuous images of a compact metric space.*

In this note it is shown that if M is a set of points and for each sequence $[q_i]$ of M there exists a "limit" function such that (1) limit $[q_i]$ is a unique subset of M (which may be vacuous), (2) if limit $[q_i] \neq 0$ and $[q_{n_i}]$ is a subsequence of $[q_i]$ then limit $[q_{n_i}] = \text{limit}[q_i]$, and further (3) if M is the continuous image of a compact metric space, then M is compact and metric. The result is a generalization of a theorem of Alexandroff (Mathematische Annalen, vol. 96, p. 562).

53. Professor W. A. Wilson: *A property of continua equivalent to local connectivity.*

This note proves the following theorem: If M is a metric space which is compact, separable, and connected, the necessary and sufficient condition for M to be locally connected at every point is that, for every pair of subcontinua A and B of M, there is a decomposition of M into two continua P and Q such that $P \cdot B = Q \cdot A = 0$. The same conclusion is valid if M is a bounded or unbounded continuum lying in a euclidean space.

54. Professor L. L. Dines: *Independent postulates for a generalized euclidean algorithm.*

In a Königsberg dissertation Plaut developed an algorithm for finding the highest common factor of a finite set of expressions of a general nature, including algebraic polynomials and linear differential and difference expressions as special cases. He based the development upon twelve postu-

lates, the mutual independence of which he said he had been unable to establish. In the present note it is shown that these postulates are not independent, and an equivalent set of ten independent postulates is presented.

55. Professor A. J. Kempner: *On the roots of algebraic equations and derived equations.*

This paper contains miscellaneous results of the following character: (1) A uniform method of derivation of a large number of known theorems such as *If* $f=0$ *(real coefficients) has all roots real and distinct, then* $f-kf'=0$ *has only real roots;* $f'\cdot f'-f\cdot f''=0$ *has only complex roots;* $f^2+k^2f'^2=0$ *has only complex roots;* $f^2-k^2f'^2=0$ *has only real roots,* and new results of related type. (2) For the Legendre polynomials P_n the equation $P_n'\cdot P_n'-P_n\cdot P_n''=0$ has all roots imaginary and of absolute value <1. A general theorem is given concerning the location of all roots of $f'\cdot f'-f\cdot f''=0$, f a polynomial. (3) A more precise form of Jensen's theorem concerning roots of $f'=0$, f a polynomial with real coefficients.

56. Professor H. E. Buchanan: *On a certain function of the masses in the three-body problem.*

In a recent paper (American Journal, vol. 50 (1928), No. 4) on the three-body problem, the writer encountered the following function of the masses: $f(m_1, m_2, m_3)\equiv(m_1+m_2+m_3)^2-27(m_1m_2+m_1m_3+m_2m_3)$. The present paper discusses this function in detail. If $f(m_1, m_2, m_3)>0$ the solutions of the differential equations for small oscillations contain only periodic terms and terms linear in t. On the other hand if $f(m_1, m_2, m_3)\leqq 0$, the positions are unstable due to the presence of terms of the form e^{at} as well as terms linear in t. If Gyldén (Bulletin Astronomique, vol. 1) and Moulton (Astronomical Journal, No. 483) are correct in their explanation of the Gegenschein it seems that there ought to be a much larger accumulation of matter near the equilateral triangle points than near the straight line positions, for the latter positions are always unstable with the exponential terms present. This paper has been accepted by the American Journal and will appear in an early number.

57. Professor H. L. Olson: *Doubly divisible quaternions.*

This paper derives a necessary and sufficient condition that a rational integral quaternion a be both a left and a right factor of a rational integral quaternion x.

58. Mr. J. R. Carson: *The asymptotic solution of an operational equation.*

This paper deals with the asymptotic solution of a specific operational equation with particular reference to the order of magnitude of the error committed by stopping with the numerically smallest term of the series, and the effect on numerical accuracy of adding an additional exponential term. The paper closes with a brief discussion of the general operational equation

and the conditions under which its divergent solution, where such exists, conforms to Poincaré's definition of an asymptotic series.

59. Professor C. H. Rowe: *A kinematical treatment of some theorems on normal rectilinear congruences.*

The author considers a two-parameter motion of a straight line l and a two-parameter motion of a rectangular trihedral, or observer, T, which is in one-to-one correspondence with that of l. He studies the relationship between the *actual* congruence and the *apparent* congruence which l describes, that is the congruence actually described by l and the congruence which l appears to describe to the observer T, and establishes a sufficient condition that the actual congruence be normal when the apparent congruence is normal. By means of this condition, he proves a number of known theorems and obtains extensions of certain of them.

60. Professor R. G. Lubben: *Concerning limiting sets.*

In this paper the following theorems are proved: (1) In a space L which has the distributive property, every uncountable point set has a limit point. (2) In a space S which has the distributive property, every uncountable point set *contains* a limit point of itself. (3) In order that a metric space have the distributive property, it is necessary and sufficient that it be locally compact and separable.

R. G. D. RICHARDSON,
Secretary

THE BOULDER COLLOQUIUM

The lectures delivered by Professor R. L. Moore were remarkably well attended, there being a total of 91 persons registered for the series. Since it was thought advisable to dispense with the Friday afternoon session, four lectures of one and one-fourth hours each were given instead of the five one-hour lectures as announced in the program. The subject of the Colloquium, *Point set theory*, being one with which a considerable number of American mathematicians were not familiar, advantage was taken by many of these of this opportunity to gain an insight into some of the underlying concepts of this most fascinating and fundamental branch of mathematics.

Inasmuch as it is expected that the lectures will appear in full in the Society's Colloquium series within approximately twelve months, only a brief sketch of the material covered is here given. A good idea of their organization and content can be obtained from the synopsis furnished by Professor Moore, although some deviations from the synopsis were made. The treatment of the subject, based on a set of axioms, which was given, is to a large extent original with the lecturer, a most noteworthy contribution of this type having been published by him in the Transactions of this Society in 1916. In addition to his own work, however, the material covered embraced that of a number of other mathematicians both in this country and in Europe.

On the basis of a system of seven* axioms stated in terms of the undefined concepts *point* and *region*, it is first shown

* Although nine axioms are stated in the synopsis, it was pointed out by the lecturer that Axioms 0 and 6 are superfluous and that they are included because of their usefulness in studying spaces in which a part, but not all, of the axioms of the system are satisfied. That Axiom 8 also is redundant and is to be dropped from the system was communicated to me by Professor Moore a few days after the lectures were concluded.

that any space satisfying these axioms is homeomorphic with the surface of a sphere or with the euclidean plane according as it is or is not compact. Some of the main features of the treatment up to this point are the facts, 1) that at no point, up until the very last when coordinates are introduced, is it necessary or apparently even of any advantage to introduce or make use of the notion of distance, and, 2) that a very large body of propositions follow from very small sub-systems of the whole system of axioms, such as from Axioms 0 and 1 or from Axioms 1 and 2, for example. A surprisingly large number of very fundamental theorems can be proved on the basis of Axioms 0 and 1 alone; and although a space satisfying these axioms is necessarily metric,* the use of the notion of distance, which is foreign to the system of axioms as a whole, seems to add little if anything to the simplicity of the treatment. These propositions hold true in all spaces satisfying the reduced set of axioms on the basis of which they were proved; and since the class of such spaces obviously is larger (granting the irredundancy of the whole system, of course) than that of all spaces satisfying the whole system, the usefulness of the axiomatic method of procedure is undeniable. This fact was brought out very strikingly by the lecturer in the following manner. It was shown that on the basis of Axioms 1 and 2, it follows that every region is arcwise connected, and that the whole space, if connected at all, is arcwise connected. It was then pointed out that in any locally connected inner limiting set (that is, a set of points which is the common part of a family of open sets—frequently called a G_δ set) in a euclidean space of any number of dimensions, or indeed in much more general space, *region* may be defined in such a way that Axioms 1 and 2 are satisfied. Accordingly, it follows that *any connected and locally connected inner limiting set* (*or* G_δ *set*)

* This fact, which was unknown to the lecturer at the time of the lectures, was communicated to me by him less than a week after their conclusion.

is arcwise connected. This is a very interesting new result. The axiomatic method thus elucidates its usefulness not only as a very direct and easy approach to the subject, which it certainly is, but also in the approach to new problems through the elimination of extraneous material which adds merely to the confusion of the worker.

The lecturer then pointed out that the whole system of axioms is satisfied in a space whose elements are the elements of any upper semi-continuous collections of mutually exclusive compact continua filling up the surface of a sphere or the plane and none separating the sphere or plane. Accordingly, such a space is equivalent to the sphere or plane respectively. Applications of this extremely useful idea were discussed, as were also applications of upper semi-continuous collections and upper semi-continuous decompositions in the study of the structure of continuous curves and continua in general.

G. T. WHYBURN

AN INTRODUCTION TO THE THEORY OF NUMBERS*

BY G. H. HARDY

PART I

1. *Farey Series.* The theory of numbers has always occupied a peculiar position among the purely mathematical sciences. It has the reputation of great difficulty and mystery among many who should be competent to judge; I suppose that there is no mathematical theory of which so many well-qualified mathematicians are so much afraid. At the same time it is unique among mathematical theories in its appeal to the uninstructed imagination and in its fascination for the amateur. It would hardly be possible in any other subject to write books like Landau's *Vorlesungen* or Dickson's *History*, six great volumes of overwhelming erudition, better than the football reports for light breakfast table reading.

The excursions of amateur mathematicians into mathematics do not usually produce interesting results. I wish to draw your attention for a moment to one very singular exception. Mr. John Farey, Sen., who lived in the Napoleonic era, has a notice of twenty lines in the *Dictionary of National Biography*, where he is described as a geologist. He received as a boy "a good mathematical training". He was at one time agent to the Duke of Bedford, but afterwards came to London, where he acquired an extensive practice as a consulting surveyor, which led him to travel much about the country and "collect minerals and rocks". His principal work was a geological survey of Derbyshire, undertaken for the Board of Agriculture, but he also wrote papers in the *Philosophical Magazine*, on geology and on many other subjects, such as

* The sixth Josiah Willard Gibbs Lecture, read at New York City, December 28, 1928, before a joint session of the American Mathematical Society and the American Association for the Advancement of Science.

music, sound, comets, carriage wheels and decimal coinage. As a geologist, Farey is apparently forgotten, and, if that were all there were to say about him, I doubt that he would find his way into the *Dictionary of National Biography* today.

It is really very astonishing that Farey's official biographer should be so completely unaware of his subject's one real title to fame. For, in spite of the *Dictionary of National Biography*, Farey is immortal; his name stands prominently in Dickson's *History* and in the German encyclopaedia of mathematics, and there is no number-theorist who has not heard of "Farey's series". Just once in his life Mr. Farey rose above mediocrity and made an original observation. He did not understand very well what he was doing, and he was too weak a mathematician to prove the quite simple theorem he had discovered. It is evident also that he did not consider his discovery, which is stated in a letter of about half a page, at all important; the editor of the Philosophical Magazine printed a very stupid criticism in the next volume, and Farey, usually a rather acrid controversialist, ignored it completely. He had obviously no idea that this casual letter was the one event of real importance in his life. We may be tempted to think that Farey was very lucky; but a man who has made an observation that has escaped Fermat and Euler deserves any luck that comes his way.*

Farey's observation was this. The *Farey series of order* n is the series, in order of magnitude, of the irreducible rational fractions between 0 and 1 whose denominators do not exceed n. Thus

$$\frac{0}{1}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2},$$

$$\frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{1}{1}$$

* It should be added that Farey's discovery had been anticipated 14 years before by C. Haros: see Dickson's *History*, vol. 1, p. 156. Cauchy happened to see Farey's note and attributed the theorem to him, and everyone else has followed Cauchy's example.

is the Farey series of order 7. There are two simple theorems about Farey series; (1) if p/q and p'/q' are two consecutive terms, then

$$p'q - pq' = 1,$$

and (ii) if p/q, p'/q', p''/q'' are three consecutive terms, then

$$\frac{p'}{q'} = \frac{p + p''}{q + q''}.$$

The second theorem (which is that actually stated by Farey) is an immediate consequence of the first, as we see by solving the equations

$$p'q - pq' = 1, \quad p''q' - p'q'' = 1,$$

for p' and q'.

The theorems are not of absolutely first class importance, but they are not trivial, and all of the many proofs have some feature of real interest. One of the simplest uses the language of elementary geometry. We consider the *lattice* or *Gitter* L in a plane formed by drawing parallels to the axes at unit distance from each other; the intersections, the points (x, y) with integral coordinates, are called the *points of the lattice*. It is obvious that the properties of the lattice are independent of the particular lattice point O selected as origin and symmetrical about any origin. The lattice is transformed into itself by the linear substitution

$$x' = \alpha x + \beta y, \quad y' = \gamma x + \delta y,$$

where α, β, γ, δ are integers and $\Delta = \alpha\delta - \beta\gamma = 1$, since then there is a pair x, y which give any assigned integral values for x', y'.

The area of the parallelogram P based on the origin and two lattice points (x_1, y_1), and (x_2, y_2), not collinear with O, is

$$\delta = \pm (x_1y_2 - x_2y_1).$$

We can construct a lattice L' (an oblique lattice) by producing and drawing parallels to the sides of P. A necessary and sufficient condition that L' should be equivalent to L, that

is, that they should contain the same lattice points, is that $\delta=1$, that is, that δ should have its smallest possible value. It is clear that this is also a necessary and sufficient condition that *there should be no lattice point inside* P, and it is easy to see that if there is such a point inside P, there is one inside, or on the boundary of, the triangular half of P nearer to O.

We may call the lattice point (q,p) which corresponds to a fraction p/q in its lowest terms a *visible* lattice point; there is no other lattice point which obscures the view of it from O. Let us consider all the visible lattice points which lie inside, or on the boundary of, the triangle bounded by the lines $y=0$, $x=n$, $y=x$. It is plain that these points correspond one by one to the fractions of the Farey series of order n. When the ray R from O to (q, p) rotates from the x-axis to the line $y=x$, it passes through each of these points in turn. If we take two consecutive positions of R, corresponding to the points (q, p), (q', p'), the parallelogram based on these two points contains no lattice point inside it, since otherwise there would be a lattice point inside its nearer triangle, and therefore a Farey fraction between p/q and p'/q'. It follows that

$$\delta = p'q - pq' = 1,$$

which proves Farey's theorem.

2. *Purpose of this Lecture.* So much then for Farey's discovery; it is a curious theorem, and its history is still more curious; but I have no doubt allowed myself to dwell upon it a little longer than its intrinsic importance deserves. My discussion of it will, however, help me to explain what I am trying to do in this lecture.

I shall imagine my audience to be made up entirely of men like Farey. I know that most of them are very much better mathematicians, but I shall not assume so; I shall assume only that they possess the common school knowledge of arithmetic and algebra. But I shall also assume

that, like Farey, they are curious about the properties of integral numbers; one need after all be no Ramanujan for that.

Let us then imagine such a man playing about with numbers (as so many retired officers in England do) and puzzling himself about the curious properties which they seem to possess. What odd properties would strike him? What are the first questions he would ask? We must not try to be very systematic; if we do, we shall make no progress in an hour. We must aim merely at a rough preliminary survey of the ground. If in the course of our survey, we find the opportunity for any illuminating remark, we may delay to make it, as I have already delayed over Mr. Farey, even if it does not seem to fall in quite its proper logical place. Then, if time permits, we may return to examine a little more closely any important difficulties which our preliminary survey has revealed.

3. *Congruences to a Modulus.* There is no doubt that the first general idea which we should have to explain is that of a *congruence.* Two numbers a and b are *congruent to modulus* m if they leave the same remainder when divided by m, that is, if m is a divisor of $a-b$. We write

$$a \equiv b \pmod{m}, \quad m \mid a - b.$$

It is obvious that congruences are of immense practical importance. Ordinary life is governed by them; railway time tables and lists of lectures are tables of congruences. The absolute values of numbers are comparatively unimportant; we want to know what time it is, not how many minutes have passed since the creation.

A great many problems both of arithmetic and of common life depend upon the solution of congruences involving an unknown x, such as

$$a_0 x^n + a_1 x^{n-1} + \cdots + a_n \equiv 0 \pmod{m}.$$

Such congruences may be classified like algebraical equations, as linear, quadratic, $\cdots$, according to the value of n.

Our first instinct in dealing with congruences is to follow up the analogy with algebra. In algebra a linear equation has one root, a quadratic two, and so on. We find at once that there are obvious and striking contrasts; even the linear congruence suggests a whole series of problems, and a full discussion of quadratic congruences involves quite an imposing body of general ideas.

Let us take the simplest case, the linear congruence, and suppose first that we are concerned only with one particular modulus, such as 7 or 24. We have then an example of a genuinely finite mathematics. Congruent numbers have exactly the same properties and cannot be distinguished, and our mathematics contains only *a finite number of things*. In such a mathematics any problem can be solved by enumeration; we can solve $2x \equiv 5 \pmod 7$ by trying all possible values of x, and we find there is a unique solution, $x \equiv 6$. If we try to solve $2x \equiv 5 \pmod{24}$, we find that there is no solution; if I lecture every other day, I shall sooner or later lecture on Thursday, but if I lecture every other hour, I may never lecture at 5 P.M.

The difference is of course accounted for by the fact that 7 is *prime* and 24 is not. Here we encounter the notion of a prime, a number without factors, and all kinds of speculations suggest themselves. Can we tell, by any method short of trial of all possible divisors, whether any given number is prime or not? Are there formulas for primes? Are the primes infinite in number, and if so, what is the law of their distribution?

Again, it appears that all numbers are composed of primes, that primes are the ultimate material out of which the world of numbers is built up. We are bound to ask *how;* and here we meet our first big theorem, the "fundamental theorem of arithmetic," the theorem that factorization is unique. But we shall probably be wise to allow our enquirer to take this theorem for granted until he has acquired a little of the sophistication which comes with wider knowledge.

We may observe, however, before passing on, that the con-

trast between arithmetic and algebra becomes much more marked as soon as we consider congruences of higher degree. An equation of the fourth degree has, with appropriate conventions, just four roots. But

$$x^4 \equiv 1 \pmod{13}$$

has 4 roots, 1, 5, 8, and 12;

$$x^4 \equiv 1 \pmod{16}$$

has 8 roots, 1, 3, 5, 7, 9, 11, 13, and 15; and

$$x^4 \equiv 2 \pmod{16}$$

has none.

4. *Regarding Decimals.* I pass to another subject that has an irresistible fascination for amateurs, the subject of *decimals*. Some decimals are finite and some recurring, but it is easy to write down decimals, such as

(a) $0.10100100010\cdots$ (b) $0.11010001000\cdots$

which are neither. Here (a) the number of 0's increases by one at each stage, (b) the ranks of the 1's are $1, 2, 4, 8, \cdots$. More amusing examples are

(c) $0.01101010001010\cdots$

(in which the 1's have prime rank) and

(d) $0.23571113171923\cdots$

(formed by writing down the prime numbers in order). The proof for (c) demands the knowledge that there is an infinity of primes, and that for (d) rather more.*

The answer to some of the obvious questions is immediate. A finite decimal represents a rational fraction $p/(2^\alpha 5^\beta)$, a pure recurring decimal a fraction p/q, where q is not divisible by 2 or 5, and a mixed recurring decimal a fraction in which q is divisible by 2 or 5 and also by some other number. The converses of these theorems are also true, but the proof demands a little genuinely arithmetical reasoning. I shall state the proof in the simplest case, since it depends upon

* See Pólya and Szegö's *Aufgaben aus der Analysis*, vol. 2, pp. 160, 383.

the logical principle which is perhaps our most effective weapon in the elementary parts of the theory, where we are dealing with so simple a subject matter that our choice of arguments is naturally very restricted.

Suppose $p<q$ and q prime to 10. If we divide all powers 10^r by q, there are only q possible remainders, and one at least must be repeated. It follows that there are a ν_1 and a $\nu_2>\nu_1$ such that

$$10^{\nu_2} \equiv 10^{\nu_1}, \quad 10^{\nu_1}(10^{\nu_2-\nu_1} - 1) \equiv 0$$

to modulus q. It follows that, if we write $\nu_2-\nu_1=N$, we have $10^N \equiv 1$, so that $q \mid 10^N-1$ and

$$\frac{p}{q} = \frac{P}{10^N - 1} = P\cdot 10^{-N} + P\cdot 10^{-2N} + \cdots .$$

Since $P<10^N$, this is a pure recurring decimal with a period of at most N. The principles which we have used are (a) that *if there are more than q things of at most q kinds, there must be two of them of the same kind*; (b) that if 10^rQ is divisible by q, and q is prime to 10, then Q is divisible by q. In the second we are of course appealing to the "fundamental theorem". The first is the general logical principle to which I referred just now.

Let us take a slightly more complicated variant of this principle. *If there are two sets of objects*

$$a_1, a_2, \cdots, a_m, \quad b_1, b_2, \cdots, b_m,$$

no two of either set being the same; and if every b is equal to an a; then the b's are the a's arranged in a different order. We may apply this principle to obtain further information about the period of our recurring decimal. I suppose now that q is prime. If q and a are given, and a is not a multiple of q, it is impossible that

$$ra \equiv sa \pmod{q}$$

unless $r \equiv s$. If (ra) is the remainder when ra is divided by q, the two sets

$$r, \quad (ra) \qquad (r = 1, 2, \cdots, q-1)$$

satisfy the conditions of our principle and are therefore the same except in order. It follows that

$$(q-1)!\,a^{q-1} \equiv \prod(ra) \equiv \prod r = (q-1)! \pmod{q},$$

and therefore that

$$a^{q-1} \equiv 1 \pmod{q};$$

Fermat's Theorem. In the particular case in which we are interested, a is 10, and Fermat's Theorem shows that we may take $N=q-1$, so that the period of p/q cannot exceed $q-1$ figures. Observe that we have appealed to the fundamental theorem twice in the proof.

It is familiar to everyone that $\frac{1}{7}$ has 6 figures, the maximum number. We are bound to ask what other primes q possess this property; the values of q less than 50 are in fact 7, 17, 19, 23, 29, and 47, but here we begin to get into deeper water. I cannot stop to discuss this question now, but before passing on I must mention another familiar text-book theorem which I shall have to quote later. This is Wilson's Theorem, that

$$(q-1)! + 1 \equiv 0 \pmod{q}$$

if and only if q is prime. Of the mass of proofs catalogued by Dickson, that of Dirichlet depends most directly on principles which we have used already. It is an immediate consequence of these principles that, if x is any one of the set $1, 2, \cdots, q-1$, there is just one other, y, such that $xy \equiv 1 \pmod{q}$; we call y the *associate* of x. It is plain that 1 and $q-1$ are associated with themselves; and no other number can be, since $x_1^2 \equiv x_2^2$ implies $x_1 \equiv x_2$ or $x_1 \equiv q - x_2$. It follows that the numbers $2, 3, \cdots, q-2$ are composed of $\frac{1}{2}(q-3)$ distinct pairs the product of each of which is congruent to 1. Hence

$$2 \cdot 3 \cdots (q-2) \equiv 1^{(q-3)/2} = 1,$$
$$(q-1)! \equiv q - 1 \equiv -1,$$

which is one half of Wilson's Theorem. The converse half is practically obvious, since $(q-1)!$ would be divisible by any factor of q.

5. *Algebraic and Transcendental Numbers.* The study of decimals leads directly to problems concerning *rationality and irrationality.* Our decimals such as $0.1010010001\cdots$ must represent irrational numbers. What criteria are there for deciding whether a given number is rational or irrational? To ask this question is to go a little outside the theory of numbers proper, which is concerned first with integers, and then with rationals or irrationals of special forms, such as the form $a+b\sqrt{2}$, and not with irrationals as a whole or general criteria for irrationality. The problem is, however, one about which an amateur will certainly demand information.

The famous argument of Pythagoras shows that $\sqrt{2}$ is irrational; if a/b is in its lowest terms and $a^2=2b^2$, then a and b must both be even, a contradiction. It is obvious to us now that the Pythagorean argument extends at once to $\sqrt{3}$, $\sqrt{5}$, $\cdots$, $2^{1/3}$, $\cdots$, and generally to $N^{1/m}$, where N is any number which is not a perfect mth power. There is a curious and very instructive historical puzzle connected with this argument. There is a passage in Plato's *Theaetetus*, discussed at length by Heath in his *History of Greek Mathematics*, about the attempt of Theodorus to generalize Pythagoras's proof. Theodorus, working some 50 years after Pythagoras, proved the irrationality of $\sqrt{N}$ for all values of N (except square values) up to 17 inclusive. Why, ask the historians did he stop? Why in any case should it have taken mathematicians like the Greeks 50 years to make so obvious an extension? Zeuthen in particular expended a great deal of ingenuity upon this question, but I think that the ingenuity was misplaced, and that the answer is obvious.

Theodorus *did not know the fundamental theorem of arithmetic;* there is something of a puzzle about the history of that theorem, but it cannot have been known to the Greeks before Euclid's time. The triviality of the generalization to us is due entirely to our knowledge of this theorem. Suppose, for example, we wish to prove that

$$a^2 = 60b^2,$$

where a and b are integers without common factor, is impossible. We argue that a^2 cannot be divisible by 3 unless a is divisible by 3; hence $a = 3c$, $a^2 = 9c^2$, $3c^2 = 20b^2$, and a repetition of the argument shows that b also is divisible by 3. We can prove that $3 \mid a^2$ implies $3 \mid a$ *without the fundamental theorem*, by enumeration of possible cases, considering separately the cases in which $a \equiv 0, 1, 2 \pmod 3$. If it were 17 instead of 3, the process would be a little tedious; and in any case such a classification of numbers would have been very novel in Theodorus's time. I am so far from being puzzled by the limitations of his work that I regard what he did as a very remarkable achievement.

There are very few types of numbers which present themselves at all naturally in analysis and which can be proved to be irrational. It is obvious that a number like $\log_{10} 2$ is irrational, for a power of 2 cannot be a power of 10. The proof for e, from the exponential series, is quite easy, and that for e^2 not very much more difficult. That for π is decidedly more so, and when we come to numbers like e^3 and π^2, it ceases to be worth while to worry about elementary proofs; we may as well go the whole way and prove e and π are transcendental. The most famous constant in analysis, after e and π, is Euler's constant γ; and the proof of the irrationality of γ is one of the classical unsolved problems of mathematics. It has never been proved that $2^{\sqrt{2}}$, $3^{\sqrt{2}}$, and similar numbers are irrational; no plausible method for attacking such problems has even been suggested. I am inclined to think that the number which holds out the best hopes for new discovery is the number e^{π}, which presents itself so naturally in the formulas of elliptic functions.

I said just now that e and π were "transcendental". I must not stop to talk at length about this famous theorem of Lindemann,* which contains the final proof that the

* See for example Hobson's *Trigonometry*, third edition, p. 305, or the same author's *Squaring the Circle*.

quadrature of the circle, in the classical sense, is impossible; but the *statement* of the theorem introduces a notion that we shall require, that of an *algebraic number*. An algebraic number is the root of an equation

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = 0,$$

where the a's are integers. An *algebraic integer* is an algebraic number whose characteristic equation has unity for its leading coefficient. Thus $\sqrt{2}$ and $1+\sqrt{(-5)}$ are algebraic integers. A *transcendental* number is a number which is not algebraic; and Lindemann's Theorem is that π *is transcendental*. It is easy to show that all lengths which can be constructed by euclidean methods are algebraic, and indeed algebraic numbers of a quite special kind. If follows that the quadrature of the circle by any euclidean construction is impossible.

There is another direction in which we may be tempted to digress at this point, the theory of the approximation of irrationals by rationals, what is now called "diophantine approximation". There is just one theorem in this field that I shall mention, because it is connected so directly with what I have just been saying, and because it depends upon another of the stock arguments of number theory, the principle that *an integer numerically less than* 1 *is* 0. This is Liouville's theorem, that *there are transcendental numbers*. It is naturally much easier to prove this than to prove that a given number such as π is transcendental.

Liouville proves first that *it is impossible to approximate rationally to an algebraic number with more than a certain accuracy*. It is quite easy to see why. Suppose that ξ is an algebraic number defined by

$$f(\xi) = a_0\xi^n + a_1\xi^{n-1} + \cdots + a_n = 0.$$

We may suppose that the equation is irreducible, that is to say that $f(\xi)$ cannot be resolved into simpler algebraic factors of similar form; in this case we say that ξ is *of degree* n. We can obviously find a number M, depending only on ξ,

such that

$$|f'(x)| < M$$

for x near ξ. Suppose now that p/q is a rational, near ξ. Then

$$f\left(\frac{p}{q}\right) = \frac{N}{q^n},$$

where N is an integer not zero. It follows from our general principle that $|N| \geqq 1$ and

$$\left| f\left(\frac{p}{q}\right) \right| \geqq \frac{1}{q^n}.$$

But

$$f\left(\frac{p}{q}\right) = f\left(\frac{p}{q}\right) - f(\xi) = \left(\frac{p}{q} - \xi\right) f'(\eta),$$

where η lies between p/q and ξ. Hence, for all q,

$$\left| \frac{p}{q} - \xi \right| = \left| \frac{f(p/q)}{f'(\eta)} \right| > \frac{1}{Mq^n}.$$

It is impossible to approximate rationally to an algebraic number of degree n with an order of accuracy higher than q^{-n}.

On the other hand it is easy to write down numbers which have rational approximations of much higher accuracy than this; we have only to take a decimal of 0's and 1's in which the 1's are spaced out sufficiently widely. Thus

$$\xi = \frac{1}{10^{1!}} + \frac{1}{10^{2!}} + \frac{1}{10^{3!}} + \cdots = .11000100000\cdots$$

is approximated by its first k terms, that is, by a fraction

$$\frac{p}{q} = \frac{p}{10^{k!}}$$

with an error of order $10^{-(k+1)!} = q^{-k-1}$. Hence it is not an algebraic number of degree k and, since k is arbitrary, it must be transcendental. Obviously Liouville's argument enables us to *construct* transcendental numbers as freely as we please.

6. *Arithmetic. Forms.* The theory of irrationals starts from Pythagoras, and there is another great branch of the theory of numbers which also starts from him and about which I must now say something. This is the theory of *forms*.

Our interest in the theory of forms begins when we observe that there are Pythagorean triangles with integral sides; thus $3^2+4^2=5^2$. The first problem which suggests itself is that of determining all such triangles, and the solution given in substance by Diophantus, is easy. All the integral solutions of

$$x^2 + y^2 = z^2$$

are given by

$$x = \lambda(\xi^2 - \eta^2), \qquad y = 2\lambda\xi\eta, \qquad z = \lambda(\xi^2 + \eta^2),$$

where the letters are integers and ξ and η are coprime and of opposite parity. This problem is trivial, but it suggests an infinity of others.

It is natural to begin by a generalization of the problem. Let us discard the hypothesis that the hypotenuse z is integral; then

$$n = z^2 = x^2 + y^2$$

is the sum of two squares, and we are led to ask what numbers n possess this property. This is the first and simplest problem in the theory of *quadratic forms*, and the answer to it shows that no such problem can be quite easy. Even linear forms are not quite trivial; the solution of $ax+by=n$ in integers is a quite interesting elementary problem. When we consider quadratic forms, we come up against difficulties of a different order.

The first theorem in the subject is another theorem of Fermat, that *$x^2+y^2=n$ is soluble when n is a prime $p=4m+1$* and, apart from trivial variations of the sign and order of x and y, uniquely. It is to be observed that the equation is plainly insoluble when n is $4m+3$, since any square is congruent to 0 or 1 to modulus 4. This theorem is one of the most famous in the theory of numbers, and very rightly so, since it was the first really difficult theorem in the subject

proved by any mathematician. There is no really simple proof, and the most natural, that which depends on the Gaussian numbers $a+bi$, introduces a whole series of ideas of revolutionary importance.

The first stage of the proof consists in proving that *there is a number x such that*

$$x^2 \equiv -1 \pmod{p},$$

or $p|1+x^2$. Let us go back for a moment to the proof I sketched of Wilson's Theorem. Let us associate the numbers $x=1, 2, \cdots, p-1$ in pairs x, y not, as then, so that $xy \equiv 1$, but so that

$$xy \equiv -1 \pmod{p}.$$

If any x is associated with itself, our proposition is established. If not, we have arranged the numbers from 1 to $p-1$ in $\frac{1}{2}(p-1)$ pairs of different numbers each satisfying the condition. Hence

$$(p-1)! \equiv \prod xy \equiv (-1)^{(p-1)/2} = 1;$$

which is false, since, by Wilson's Theorem,

$$(p-1)! \equiv -1.$$

We thus obtain our proposition by reductio ad absurdum.

The second stage of the proof depends on much more novel ideas. We are concerned with the simplest case of an *algebraic field*. The field $K(i)$ is the aggregate of numbers

$$\xi = r + si = r + s\sqrt{(-1)},$$

where r and s are rational. This number satisfies the equation

$$\xi^2 - 2r\xi + r^2 + s^2 = 0,$$

and is an algebraic integer, in the sense I defined before, when $2r$ and r^2+s^2 are integers, that is, when r and s are integers. We may denote by $K^*(i)$ the aggregate of all the integers

$$\alpha = a + bi$$

of $K(i)$; a and b are ordinary integers. The numbers of $K^*(i)$ reproduce themselves by addition and multiplication, and we can define division in this field just as we define it in ordinary arithmetic. We can also define a *prime* of $K^*(i)$, and factorization of numbers into primes. There are four numbers, ± 1 and $\pm i$, which play a part in the new arithmetic similar to that of 1 and -1 in ordinary arithmetic. These are the "unities" or divisors of 1. If we define the *norm* of $\alpha = a + bi$ as

$$N(\alpha) = a^2 + b^2,$$

then the unities are characterized by the fact that their norm is 1. We do not count them as primes, just as, in the ordinary theory, we do not count 1 as a prime.

We now make an assumption, namely that *the analog of the fundamental theorem holds in the field* $K^*(i)$, that is to say that, apart from any trivial complications which may be introduced by the unities, *the factorization of a number of* $K^*(i)$ *into primes is unique.* This assumption is in fact correct. Returning now to the first stage of our proof, there is an x such that

$$p \mid 1 + x^2 = (1 + ix)(1 - ix).$$

It is obvious that p does not divide $1+ix$ or $1-ix$, so that *p divides the product of two numbers without dividing either of them.* Hence p cannot be a prime in $K^*(i)$. We may therefore write

$$p = \pi\lambda,$$

where $N(\pi) > 1$ and $N(\lambda) > 1$. But

$$N(\pi)N(\lambda) = N(p) = p^2,$$

so that $N(\pi)$ and $N(\lambda)$ must each be p. If we write

$$\pi = a + ib,$$

it follows that

$$p = N(\pi) = a^2 + b^2,$$

which is Fermat's theorem.

We may be tempted by our success to further efforts in the same direction. It is easy to satisfy ourselves, by considering particular cases, that *any prime* $p=20m+1$ *is of the form* a^2+5b^2: thus $61=4^2+5\cdot 3^2$. Let us try to prove this theorem by a similar method. We must evidently consider now the field $K^*[\sqrt{(-5)}]$ formed of the algebraic integers of the form

$$\alpha = a + b\sqrt{(-5)}\ ;$$

it is easy to show that such a number is an algebraic integer if and only if a and b are ordinary integers. There is no difficulty in defining divisibility and primality in this field also.

The first step in our proof must plainly be to prove the existence of an x for which $p \mid 1+5x^2$. This is not difficult, but it demands a little more knowledge of quadratic congruences than I can assume, and I must take it for granted.

We define the norm $N(\alpha)$ of a number of this field as a^2+5b^2. We then argue as before; we have

$$p \mid 1 + 5x^2 = (1 + x\sqrt{(-5)})(1 - x\sqrt{(-5)}),$$

so that p divides a product without dividing either factor and is therefore not a prime. Hence, as before $p=\pi\lambda$, where $N(\pi)>1$ and $N(\lambda)>1$, and $N(\pi)$ and $N(\lambda)$ must each be p. It follows that

$$p = N(\pi) = a^2 + 5b^2,$$

the theorem we set out to prove.

At this point, however, there is a shock in store for us; we find that we can prove *too much*. The number

$$q = (2 + \sqrt{(-5)})(2 - \sqrt{(-5)})$$

is divisible by 3, while neither factor is so. Hence 3 is not a prime. Hence

$$3=\pi\lambda, \quad 9=N(\pi)N(\lambda),$$

and $N(\pi)$ and $N(\lambda)$ are each 3. It follows that

$$3 = N(\pi) = a^2 + 5b^2.$$

Similarly we can prove that

$$7 = a_2 + 5b_2;$$

and both of these theorems are obviously false.

There must therefore be a mistake somewhere in our argument, and if you examine it, and are prepared to believe that I have not been misleading you wilfully, you will see that there is only one step which can be questioned. In all three cases I concluded the argument by an appeal to the same theorem; *a number which divides the product of two numbers without dividing either of them cannot be prime.* This is true in ordinary arithmetic, because of the fundamental theorem; if 7 were a divisor of $15 = 3 \cdot 5$, 15 would be factorable into primes in two distinct manners. It follows that *the analog of the fundamental theorem in the field* $K^*[\sqrt{(-5)}]$ *must be false*; and this is easily verified when once our suspicions have been excited; thus

$$2 \cdot 3 = (1 + \sqrt{(-5)})(1 - \sqrt{(-5)}),$$

$$3 \cdot 7 = (1 + 2\sqrt{(-5)})(1 - 2\sqrt{(-5)}),$$

and all of these numbers are prime in $K^*[\sqrt{(-5)}]$. The proof which I gave of the theorem concerning primes $20m+1$ was therefore fallacious, although the theorem is true. The proof of Fermat's theorem, on the other hand, was correct, since factorization *is* unique in $K^*(i)$.

7. *Further Problems.* It is clear that we must go back to the beginning and study the theory of primes a little more closely; but before I do this I should like to call your attention to a series of further problems suggested by Fermat's theorem. We know now when a *prime* is the sum of two squares, and we have to consider the same problem for general n. Here in fact there are three different problems.

The first and most obvious problem is that of determining the necessary and sufficient conditions that n should be representable. This problem may be solved quite easily with the aid of the Gaussian numbers; n must be $2^\alpha M^2 N$, where α is 0 or 1 and N contains prime factors of the form

$4m+1$ only. We are then led naturally to the corresponding problem for other forms, first for the general binary quadratic form

$$ax^2 + bxy + cy^2,$$

then for quadratic forms in a larger number of variables, such as

$$x^2 + y^2 + z^2, \quad x^2 + y^2 + z^2 + t^2,$$

and then for forms of higher degree, such as x^3+y^3 and x^4+y^4. There is a highly developed theory of the general quadratic form; the most famous theorem is perhaps Lagrange's theorem, that *every number is the sum of four squares.* But as soon as we begin to consider cubic or higher forms we find ourselves on the boundary of knowedge. There is for example no criterion analogous to Fermat's by which we can decide whether a given number is the sum of two cubes.

The second problem about the form x^2+y^2 suggested by Fermat's theorem is that of determining the *number of representations.* This problem may be interpreted in two different ways. We may want an exact formula, in terms of the factors of n, and in this case the Gaussian theory again gives what we want; $r(n)$, the number of representations, is given by the formula

$$r(n) = 4\{d_1(n) - d_3(n)\},$$

where $d_1(n)$ and $d_3(n)$ are the numbers of divisors of n of the forms $4m+1$ and $4m+3$ respectively. This is, however, not the most interesting interpretation of the problem. We may want, not a formula like this, but information concerning the *order of magnitude* of $r(n)$, whether $r(n)$ is generally large when n is large, whether numbers are usually representable freely or with difficulty. In this case our formula gives us very little help, and the solution of the problem requires quite different methods.

It is here that we come into contact for the first time with a new branch of the theory, the modern "analytic" theory.

This theory has two special characteristics. The first is one of method; it uses, besides the methods of the classical theory, the methods of the modern theory of functions of a complex variable. The second is that it is concerned primarily with problems of order of magnitude and asymptotic distribution. The distinction is not a perfectly sharp one; there are "exact", "finite" theorems which have only been solved by "analytic" methods. For example, *every number greater than* $10^{10^{10}}$ *is expressible as the sum of* 8 *cubes*; this theorem includes no reference to "order of magnitude", and is a "finite" theorem in just the same sense as Fermat's theorem about the squares, but the only known proof is analytic. On the whole, however, it is the problems of asymptotic distribution which dominate the theory.

The answer given by the analytic theory to the special question which I raised is roughly as follows. The average value of $r(n)$ is π. It must be observed that representations which differ only trivially, that is, in the sign or order of x and y, are reckoned as distinct. If we allow for this, the average number of representations is rather less than a half; this is explained by the fact that, as we shall see, *most* numbers are not representable. On the other hand $r(n)$ tends to infinity with n with tolerable rapidity for numbers of appropriate forms, more rapidly for example than any power of $\log n$. The corresponding problems for cubes or higher powers present difficulties which are at present quite insuperable, and all that I can do is to mention a few curiosities. The smallest number representable by two cubes in two really distinct ways is

$$1729 = 1^3 + 12^3 = 9^3 + 10^3,$$

and the smallest representable in three ways is probably

$$175959000 = 70^3 + 560^3 = 198^3 + 552^3 = 315^3 + 525^3.$$

It can be proved that there exist numbers with as many different representations as we please. A. E. Western has carried out very heavy computations concerning representa-

tions by cubes; he has for example found 6 numbers, of which the smallest is 1,259,712, representable as the sum of *three* cubes in *six* different ways. The smallest number doubly representable by two fourth powers is probably

$$635318657 = 59^4 + 158^4 = 133^4 + 134^4 ;$$

there is, so far as I know, no known example of a number with three such representations, nor any proof that such a number exists.

The nature of the problems of the analytic theory becomes clearer when we consider the third problem suggested by Fermat's theorem. This is the problem of determining the *distribution* of the representable numbers. We want to know *how many numbers are representable*, or, to put it more precisely, how many numbers less than a large assigned number x are representable. If $Q(x)$ is the number of such numbers, what is the order of magnitude of $Q(x)$? Are nearly all numbers representable, or just a majority, or only a few? The answer is in fact that $Q(x)$ is approximately

$$\frac{Ax}{(\log x)^{1/2}},$$

where A is a constant; to put it roughly, quite a lot of numbers are representable, but strictly an infinitesimal proportion of the whole. This explains why the average number of representations turned out to be less than one.

This problem about $Q(x)$ is a very interesting one, but there is another of the same kind which is obviously still more interesting and much more fundamental. This is the problem of the distribution of the primes themselves; *how many primes are there less than x?* I shall say something about this problem in a moment; it is in any case time for us to return to the theory of primes, since all our enquiries have ended in questions about them, and it is obviously impossible to make serious progress until we know more both of their elementary properties and of the laws which govern their distribution.

PART II

8. *The Fundamental Theorem.* The *fundamental theorem* of arithmetic is the beginning of the theory of numbers, and it is plain that our first task must be to make this theorem secure.

There is another historical puzzle about the fundamental theorem. Who first stated the theorem, explicitly and generally? The natural answer is *Euclid*, since the *Elements* contain all the materials for the proof. Everything rests on Euclid's famous algorithm for the greatest common divisor. Given two numbers a, b, of which a is the greater, we form the table

$$a = bc + b_1, \qquad b = b_1c_1 + b_2, \qquad b_1 = b_2c_2 + b_3, \cdots$$

where $b_1, b_2, \cdots$, are the remainders in the ordinary sense of elementary arithmetic. Since

$$b > b_1 > b_2 > \cdots,$$

b_n must sooner or later be zero. The last positive remainder δ has the properties implied by the words *greatest common divisor*, and it follows from the process by which δ is formed that any number which divides both a and b divides δ.

Let us note in passing that there is an analogous process in $K^*(i)$, but that the analogy fails in $K^*[\sqrt{(-5)}]$. In ordinary arithmetic, given a and b, we can find a number congruent to a mod b and less than b. There is a similar theorem for the Gaussian numbers. Here there is no strict order of magnitude between different numbers, and we have to use the order of magnitude of their norms. Given α and β, there is a number, congruent to α mod β, whose norm is less than that of β. There is no such theorem in $K^*[\sqrt{(-5)}]$, and the process analogous to Euclid's fails.

When the existence of δ is once established, the proof of the fundamental theorem is easy. We write

$$\delta = (a, b)$$

and we say that a is prime to b when $(a, b) = 1$. The crucial lemma is that *if* $(a, b) = \lambda$ *and* $b \mid ac$, *then* $b \mid c$; in particular, *a prime cannot divide a product without dividing one or other of*

the factors. This once granted, anybody can construct the proof of the fundamental theorem for himself; and you will remember that it was just this proposition which led to our troubles in $K^*[\sqrt{(-5)}]$.

The lemma itself may be proved as follows. We construct the euclidean algorithm for a and b, with the final remainder 1. If we multiply it throughout by c, we have the algorithm for ac and bc, and the final remainder is c. It follows that

$$(ac,bc) = c.$$

Since b divides ac, by hypothesis, and also bc, it divides c.

This is Euclid's own argument, and with it he had proved what is essential in the fundamental theorem. It is a very singular thing that he should then omit to state the magnificent theorem that he has proved. He is over the line and free, but apparently disdains the formality of touching down. I do not know of any formal statement of the theorem earlier than Gauss. The substance of the theorem, however, is in the *Elements*; it was plainly unknown, as I explained before, to the Greeks from 50 to 100 years before Euclid's time; and I see no particular reason for questioning the obvious view that it is Euclid's own.

As soon as we have proved the fundamental theorem our elementary knowledge falls into line. The theory of linear congruences, the theorems of Fermat and Wilson and all their consequences, the elementary theory of decimals and of the divisors of numbers, may be developed straightforwardly and without the introduction of essentially new ideas. I can now say something about the more modern side of the theory of primes.

9. *Problems Concerning Primes*. What are the most natural questions to ask about primes? I say deliberately the most *natural*; we must remember that a natural question does not always seem, on fuller reflection, to have been a *reasonable* one. It is natural to an engineer to ask us for a finite formula for

$$\int e^{-x^2}dx,$$

or for a solution of some simple looking differential equation in finite terms. If we fail to satisfy him, it is not because of our stupidity, but because the world does not happen to have been made that way.

So, if any one asks us (1) *to give a general formula for the nth prime* p_n, a formula in the sense in which

$$p_n = n^2, \quad p_n = n^2 + 1, \quad p_n = [e^n],$$

where $[x]$ denotes the integral part of x, would be a formula, I can only reply that it is not a reasonable question. It is, I will not say demonstrably impossible, but wildly improbable, that any such formula exists. The distribution of the primes is not like what it would have to be on any such hypothesis. I should make the same reply to a good many other questions which an amateur might be likely to ask, for example if he asked me (2) *to give a rule for finding the prime which immediately follows a given prime.* It would of course be perfectly reasonable that he should press me for the reasons why I gave so purely a negative a reply. On the other hand the problem (3) *to find the number of primes below a given limit* is, if interpreted properly, an entirely reasonable and a soluble problem. The problems (4) *to prove that there are infinitely many pairs of primes differing by* 2, and (5) *to prove that there are infinitely many primes of the form* n^2+1, are also entirely reasonable, and if (as is the case) we cannot solve them, it is quite reasonable to condemn our lack of ingenuity.

10. *The Distribution of Primes.* If we wish to classify these problems and to decide which of them are reasonable and which are not, the first essential is to understand broadly the present state of knowledge about the distribution, the distribution *in the large* or *asymptotic* distribution, of the primes. It is this theory which gives the solution of problem (3).

We denote by $\pi(x)$ the number of primes not exceeding

x. The first step is to prove that (a) *the number of primes is infinite*; $\pi(x)$ *tends to infinity with* x. This is another of Euclid's great contributions to knowledge, and Euclid's proof is perhaps the classical example of proof by reductio ad absurdum. If the theorem is false, we may denote the primes by $2, 3, 5, \cdots, P$, and all numbers are divisible by one of these. On the other hand the number

$$(2 \cdot 3 \cdot 5 \cdots P) + 1$$

is obviously not divisible by any of $2, 3, \cdots, P$, and this is a contradiction.

Another very interesting proof is due to Pólya.* It is easy to see that any two of the numbers

$$2 + 1, 2^2 + 1, 2^4 + 1, \cdots, u_n = 2^{2^n} + 1$$

are prime to each other. For suppose that p is an odd prime and that $p \mid u_n$, $p \mid u_{n+k}$. Then also

$$p \mid 2^{2^{n+k}} - 1 = u_{n+k} - 2,$$

since

$$x^{2^k m} - 1$$

is algebraically divisible by $x^m + 1$, and therefore

$$p \mid u_{n+k} - (u_{n+k} - 2) = 2,$$

which is absurd. It follows that the number of primes less than u_n is at least n, and therefore that the number of primes is infinite. In fact the argument shows not merely that $\pi(x) \to \infty$ but that

$$\pi(x) > A \log \log x,$$

where A is a constant. Something in this direction, though a little less, can be proved by a refinement of Euclid's argument.

There is a third line of argument which is a little less elementary but may be made to prove a good deal more.†

* See Pólya and Szegö, loc. cit., pp. 133, 342.

† See Dickson's *History*, vol. 1, p. 414, where the proof is attributed to Auric.

If 2, 3, 5, $\cdots$, P were the only primes, then every number would be of the form

$$2^a 3^b 5^c \cdots P^k.$$

If this number is less than x, then a fortiori 2^a is less than x, so that a is less than a constant multiple of $\log x$, and the same argument applies to b, c, $\cdots$, k. The number of possible choices of a, b, $\cdots$, k is therefore less than a multiple of $(\log x)^\pi$, where π is the total number of primes. In other words the number of numbers less than x is less than

$$A(\log x)^\pi,$$

where A is a constant, and this is impossible, since x tends to infinity more rapidly than any power of $\log x$. A refinement of the argument leads to the inequality

$$\pi(x) > A\frac{\log x}{\log\log x};$$

and the underlying principle may be stated roughly thus, that *if the number of primes were finite, there would not be enough numbers to go round.*

We are still a very long way from the ultimate truth. It is in fact possible to prove, and by comparatively elementary methods, that *the order of magnitude of* $\pi(x)$ *is* $x(\log x)^{-1}$. This theorem, conjectured by Legendre and Gauss, was first proved by Tchebycheff in 1848.

There are two much earlier theorems of Euler which point in this direction. The first is the theorem that (b) *the series*

$$\Sigma\frac{1}{p}$$

extended over all prime numbers p, *is divergent.* The proof of this theorem depends upon an identity, also due to Euler, upon which the whole of the modern theory of primes is founded. The identity is

$$1^{-s} + 2^{-s} + 3^{-s} + \cdots = \sum n^{-s}$$

$$= \frac{1}{(1 - 2^{-s})(1 - 3^{-s})(1 - 5^{-s}) \cdots}$$

$$= \Pi\left(\frac{1}{1 - p^{-s}}\right)$$

and is valid for $s>1$; it is at bottom merely the analytical expression of the fundamental theorem, and its importance arises from the fact that it asserts the equivalence of two expressions of which one contains the primes explicitly while the other does not. From Euler's identity we deduce (b) roughly as follows: if $\sum p^{-1}$ were convergent, then

$$\Pi\left(\frac{1}{1 - p^{-1}}\right)$$

would be convergent, and therfore $\sum n^{-1}$ would be convergent, which is false. Of course the proof really needs a rather more careful statement.

Euler's second theorem is (c) *the quotient of* $\pi(x)$ *by* x *tends to zero*; or in symbols

$$\frac{\pi(x)}{x} \to 0,$$

or, as we write it now

$$\pi(x) = o(x).$$

The proportion of primes is ultimately infinitesimal, *"almost all" numbers are composite.* The theorem is a quite simple corollary of (b); roughly, if we remove from the numbers less than x all multiples of the primes $2, 3, \cdots, p$, other than these primes themselves, we are left something like

$$x\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right) \cdots \left(1 - \frac{1}{p}\right)$$

numbers. The product multiplying x tends to zero when

$p \to \infty$, because of (b), and from this we can deduce Euler's second theorem.

It is rather curious that, although Euler's second theorem is a corollary of the first, the lessons which we learn from the two theorems concerning the distribution of the primes have exactly opposite tendencies. The second theorem tells us that the number of primes below a given limit is *not too great*, that the primes are in the end rather liberally spaced out; it is in fact exactly equivalent to the theorem that (d) *the nth prime* p_n *has an order of magnitude greater than* n, or

$$\frac{p_n}{n} \to \infty .$$

If on the other hand the order of magnitude of p_n were *much* greater than n, if it were for example n^2 or $n^{10/9}$ or $n(\log n)^2$, then the series $\sum p_n^{-1}$ would be *convergent*, which is just what Euler's first theorem denies. What we learn from the two theorems together is something like this. If, as we hope, the true order of magnitude of p_n can be measured by some simple function $\phi(n)$, then that function must be of order higher than n, but somewhere near the boundary of convergence of the series

$$\sum \frac{1}{\phi(n)} .$$

The most obvious function which satisfies these requirements is $n \log n$, and to say that p_n is of order $n \log n$ is the same thing as to say that $\pi(x)$ is of order $x(\log x)^{-1}$. This is just what is asserted by Tchebycheff's theorem.

11. *Tchebycheff's Theorem.* The formal statement of Tchebycheff's theorem is (e) *the order of magnitude of* $\pi(x)$ *is* $x(\log x)^{-1}$; *there are constants* A *and* B *such that*

$$\frac{Ax}{\log x} < \pi(x) < \frac{Bx}{\log x} .$$

This theorem is precisely equivalent to (f) *the order of magnitude of* p_n *is* $n \log n$; *there are constants* A *and* B *such that*

$$An \log n < p_n < Bn \log n.$$

The proofs of these theorems given by Tchebycheff have been simplified a good deal by Landau, and I can give you a sketch of one half of the proof which should enable you to understand without much difficulty the general character of the whole.

We begin by replacing $\pi(x)$ by another function. We can write $\pi(x)$ in the form

$$\pi(x) = \sum_{p \leq x} 1 ;$$

count one for every prime up to x. A more convenient and really a more natural function is

$$\theta(x) = \sum_{p \leq x} \log p,$$

the logarithm of the product of all primes up to x. This function seems at first sight a more complicated function, but it is easy enough to see why it is more convenient to work with. The most natural operation to perform on primes is *multiplication*, and this is the operation which we employ in forming $\theta(x)$. It is because it is natural to multiply primes and not to add or subtract them that problems like the problem of the prime pairs $(p, p+2)$, or Goldbach's problem of expressing numbers as sums of primes, turn out to be so terribly difficult.

Since $x/x^{1-\delta}$ tends to infinity, for any positive value of δ, we may expect that nearly all the primes which contribute to $\theta(x)$ will lie in the interval $(x^{1-\delta}, x)$, so that their logarithms lie between $(1-\delta) \log x$ and $\log x$. Hence we may expect $\theta(x)$ to be very much the same function as $\pi(x) \log x$, and in fact there is no difficulty in proving that

$$\theta(x) \sim \pi(x) \log x,$$

that is, that the ratio of the two functions tends to 1. It follows that the inequalities in (e) are equivalent to

$$Ax < \theta(x) < Bx.$$

I shall sketch the proof of the second inequality, which is rather the simpler.

Suppose that x is a power of 2, say 2^m. The primes between $x/2$ and x divide $x!$ but not $(x/2)!$, so that

$$\prod_{x/2<p\leqq x} p \,\Big|\, \frac{x!}{(x/2)!(x/2)!}.$$

The expression on the right is *one term* in the binomial expansion of $(1+1)^x=2^x$, and therefore

$$\prod_{x/2<p\leqq x} p \leqq 2^x.$$

Replacing x by

$$x/2,\ x/4,\ x/8,\ \cdots$$

and multiplying the results, we find that

$$\prod_{p\leqq x} p \leqq 2^{x+x/2+x/4+\cdots} \leqq 2^{2x},$$

and

$$\theta(x) \leqq 2 \log 2 \cdot x.$$

This proves the theorem when $x=2^m$. If

$$2^m<x<2^{m+1}$$

we have

$$\theta(x) \leqq \theta(2^{m+1}) \leqq 4 \log 2 \cdot 2^m < 4 \log 2 \cdot x.$$

Hence we may take $B=4 \log 2$. The proof of the second inequality is, as I said, not quite so simple, but does not involve essentially more difficult ideas. We have thus determined the order of magnitude of $\pi(x)$ and of p_n, and it is perhaps a little astonishing that a problem which sounds so abstruse should have so comparatively simple a solution.

12. *The Prime Number Theorem.* Tchebycheff's solution of the problem is, however, one with which it is impossible to remain content for long, since the whole trend of our discussion has been to suggest that much more is true than we have proved. In fact Tchebycheff's work, fine as it is, is the record of a failure; it is what survives of an unsuccessful attempt to prove what is now called the *Prime Number Theorem.*

This is the theorem that (g) $\pi(x)$ *and* $x(\log x)^{-1}$ *are asymptotically equivalent; the ratio of the two functions tends to unity.* We express this by writing

$$\pi(x) \sim \frac{x}{\log x}.$$

The Prime Number Theorem is equivalent to

$$p_n \sim n \log n,$$

and we may express it very roughly by saying that *the odds are* $\log x$ *to* 1 *that a large number* x *is not prime.*

The Prime Number Theorem, the central theorem of the analytic theory of numbers, was proved independently by Hadamard and by de la Vallée-Poussin in 1896. The empirical evidence for its truth had for long been overwhelming, and I suppose that every number-theorist since Legendre had tried to prove it. The theorem differs from all those which I have discussed so far in that it is apparently impossible to prove it by properly elementary methods; there is no proof known which does not depend essentially on complex function theory. I do not mean to imply that there is any terrible difficulty in the proof; there are considerable difficulties of detail, but the fundamental ideas on which it depends are tolerably straightforward. They are, however, quite unlike any of those of which I have spoken, and I should require a whole lecture to explain them even to a strictly mathematical audience. Actually, a good deal more is known; it can be proved that $\pi(x)$ is approximated still more closely by the "logarithm-integral" of x,

$$\mathrm{Li}\, x = \int_2^x \frac{dt}{\log t},$$

that in fact

$$\pi(x) = \mathrm{Li}\, x + O\left\{\frac{x}{(\log x)^k}\right\}$$

for every k, the error being of lower order than the quotient of x by *any* power of $\log x$; and it is probable, though not yet proved, that the order of the error does not very materially exceed that of $\sqrt{x}$.

13. *Formulas for Primes.* I return now for a moment to a question which I discussed shortly before, the question whether it was reasonable to expect an "elementary formula" for the nth prime p_n. Let us imagine that my questioner was obstinate in his desire for such a formula; how could I refute his successive suggestions? If he suggested

$$p_n = n \log n,$$

I should have the obvious reply that $n \log n$ is not an integer. Suppose then that he modified his formula to

$$p_n = [n \log n].$$

I should reply that his formula did not agree with the known facts of the asymptotic theory. It agrees with $p_n \sim n \log n$, the first and most obvious deduction from the Prime Number Theorem itself; but the theory carries us much further; it enables us, for example, to show that

$$p_n = n \log n + n \log \log n + O(n),$$

which contradicts the formula. If, becoming more cautious, he asked me what ground I had for denying that p_n might be *some* elementary combination of

$$n, \log n, \log \log n, \cdots,$$

I should naturally find it harder to refute him, but I could advance three arguments which are enough in the aggregate

to make up a tolerably convincing case. (i) Since $\mathrm{Li}\,x$ is a very good approximation to $\pi(x)$, the inverse function $\mathrm{Li}^{-1}n$ must be a very good approximation to p_n. Now it is demonstrable that neither the logarithm integral nor its inverse* is an elementary function. It is therefore very unlikely that there should be an elementary formula for p_n. (ii) If the "elementary formula" does not involve the symbol $[\cdots]$ of the "integral part", the function which it defines will generally not be integral for integral n. If it does, it loses all its simplicity and all its plausibility. (iii) An elementary function may be expected to behave with tolerable regularity at infinity, and so may all its *differences*. Now extremely little is known about the difference $p_{n+1}-p_n$ of two successive primes, but everything that is known, or seems probable from the evidence of the tables, suggests *extreme irregularity* in its behavior. The Prime Number Theorem shows that the *average* value of $p_{n+1}-p_n$ must be $\log n$, and tend to infinity with n. On the other hand there is overwhelming evidence that the smallest possible values of $p_{n+1}-p_n$, namely, 2, 4, 6, $\cdots$, recur indefinitely. It seems practically certain, not merely that there are infinitely many prime pairs $(p, p+2)$ but that there are infinitely many triplets $(p, p+2, p+6)$, and so with any combination of successive primes that is arithmetically possible; such a combination as $(p, p+2, p+4)$ is naturally not possible, since one of these numbers must be divisible by 3. All this seems hopelessly inconsistent with the existence of such a formula as was suggested, and it is clear that speculation in this direction is a waste of time.

There are, however, questions which have a somewhat similar tendency and which cannot be dismissed so summarily. There is one, for example, mentioned in Carmichael's little book. The problem, as he states it, is "*to find a prime greater than a given prime*," which might be interpreted as

* This may be deduced from general theorems proved recently by J. F. Ritt.

meaning either "*to find an elementary function* $\phi(n)$ *such that* $\phi(n)\to\infty$ *and* $\phi(n)$ *is prime for every* n, *or for all* n *beyond a certain limit*" or as meaning "*to find an elementary function* $\phi(p)$ *such that* $\phi(p)>p$ *and* $\phi(p)$ *is prime whenever* p *is prime.*" With either interpretation, it is a reasonable challenge, and the problem has not been solved.

Let us take the first form of the problem, which is perhaps the more natural, and let us begin by demanding *less*, namely that $\phi(n)$ shall be prime only for *an infinity of values of* n. In this case the problem becomes trivial, since n is a solution, by Euclid's theorem. It is, however, very interesting to observe that even then n, and certain simple linear functions such as $4n-1$ and $6n-1$, are the *only* trivial solutions. Dirichlet proved that *any* linear function $an+b$ has the property required, provided only that b is prime to a, or in other words that *every arithmetical progression* (subject to the last reservation) *contains an infinity of primes*. This theorem is quite difficult, except in a few special cases such as those which I mentioned, and it exhausts our knowledge in this particular direction. No one has ever proved that any of the functions

$$n^2+1, \quad 2^n-1, \quad 2^n+1$$

is prime for an infinity of values of n. With functions of *two* variables we can progress a good deal farther; we know for example that every quadratic form $am^2+bmn+cn^2$ contains an infinity of primes, provided of course that a, b, c have no common factor and that $b^2\neq 4ac$, and we can study the law of their distribution.

To find a $\phi(n)$ prime for *every* n is naturally still more difficult. Here linear functions are obviously useless, and no solution of any kind is known. Fermat conjectured that

$$2^{2^n}+1,$$

is always prime, but Euler proved that this is false, since

$$2^{32}+1 = 4294967297 = 641\cdot 6700417.$$

So far as I know, no one else has ever advanced any other suggestion which is even plausible.

In view of the apparently insuperable difficulties of this problem, there is a certain interest in *negative* results. It is plain, first, that $an+b$ cannot be prime for all n, or all large n. More generally, no polynomial

$$f(n) = a_0n^k + a_1n^{k-1} + \cdots + a_k$$

can be prime for all or all large n; for if $f(m) = M$ then $f(rn+m)$ is divisible by M for all r. There are entertaining curiosities in this field; thus

$$n^2 - n + 41$$

is prime for the first 41, and

$$n^2 - 79n + 1601$$

for the first 80 values of n. It is obvious that forms like

$$a^n - 1, \quad a^n + 1$$

cannot be prime for all large n, since, for example, $a^{3m}-1$ is divisible by a^m-1, and it is natural to suppose that the same is true for

$$P(n, 2^n, 3^n, 4^n, \cdots, k^n),$$

where P is any polynomial with integral coefficients.†

14. *The Fundamental Theorem in an Algebraic Field.* I must not allow myself to succumb to the temptation of talking too long about the theory of the distribution of primes, which is after all only one chapter in arithmetic. There are other topics about which our imaginary enquirer will certainly demand more information, and of these I think one stands out; it is certain that he will want fuller explanations about the field $K^*[\sqrt{(-5)}]$ and the other algebraic fields in which the analog of the fundamental theorem fails. All ordinary arithmetic depends, it seems, upon the fundamental theorem; how then can there *be* an arithmetic in a field in

† Morgan Ward of Pasadena has found a very simple proof of this theorem.

which it is false? It would seem that the arithmetic of such a field can bear no real resemblance to ordinary arithmetic. I shall spend the rest of my time in an attempt to explain, in the very broadest outline, how order is restored.

I shall begin by quoting a remark of Hilbert which is trivial in itself but which shows us at once the direction in which we must look for a solution. Consider the numbers

$$1,5,9,13,17,21,\cdots$$

of the form $4m+1$. These numbers form a group for multiplication (though naturally not for addition), and we can define divisibility and primality in the group. The "primes" are the numbers

$$5,9,13,17,21,29,33,37,41,49,\cdots$$

which are greater than 1, of the form $4m+1$, and not decomposable into factors of this form. Thus 21, 57, 77, and 209 are "primes"; but

$$4389 = 21\cdot 209 = 57\cdot 77,$$

so that a number of the group may be resolved into "prime" factors in different ways.

In this case the solution of the mystery is obvious. The "fundamental theorem" fails *because of the absence from the group of the numbers* $4m+3$ *of ordinary arithmetic.* In fact

$$21 = 3\cdot 7, \quad 57 = 3\cdot 19, \quad 77 = 7\cdot 11, \quad 209 = 11\cdot 19$$

and

$$21\cdot 209 = (3\cdot 7)(11\cdot 19) = (3\cdot 19)(7\cdot 11) = 57\cdot 77.$$

We cannot give a proper account of the properties of the numbers $4m+1$ so long as we insist on excluding the numbers $4m+3$; *the numbers* $4m+1$ *do not form by themselves an adequate basis for arithmetic.* This observation has of course no intrinsic interest, since no reasonable person would expect that they would do so. It is trivial in itself, but it is not at all trivial in its suggestion, since it suggests that the troubles of

the field $K^*[\sqrt{(-5)}]$ may be remedied *by considering the field as part of some larger field.*

This is in fact the solution found by Kummer. We consider the field $L[\sqrt{(-5)}]$ of numbers

$$\xi = \sqrt{(a + b\sqrt{(-5)})},$$

where a and b are ordinary integers. This is only an approximate statement; we do not actually consider all such numbers, but only those satisfying certain further conditions; the greatest common divisor of a and b must be a square or five times a square, and a^2+5b^2 must be a square. The field L includes K^*. The numbers of L form a group for multiplication, and we can define divisibility and primality in the field. Finally, the analog of the fundamental theorem is valid; *factorization is unique in L.* The proof of this is quite simple, but requires a little attention to detail, and I must refer you for the details to Mordell's tract on *Fermat's Last Theorem.*

We can now give a simple account of the equations in $K^*(\sqrt{(-5)})$ which puzzled us before. Consider for example the equation

$$3\cdot 7 = (1 + 2\sqrt{(-5)})(1 - 2\sqrt{(-5)}).$$

It is easily verified that

$$3^2=(2+\sqrt{(-5)})(2-\sqrt{(-5)}),$$

$$7^2=(2+3\sqrt{(-5)})(2-3\sqrt{(-5)}),$$

$$(1+2\sqrt{(-5)})^2=-19+4\sqrt{(-5)}=-(2-\sqrt{(-5)})(2+3\sqrt{(-5)}),$$

$$(1-2\sqrt{(-5)})^2=-19-4\sqrt{(-5)}=-(2+\sqrt{(-5)})(2-3\sqrt{(-5)}).$$

Hence, if we write

$$\alpha = \sqrt{(2 + \sqrt{(-5)})}, \qquad \alpha' = \sqrt{(2 - \sqrt{(-5)})},$$

$$\beta = \sqrt{(2 + 3\sqrt{(-5)})}, \qquad \beta' = \sqrt{(2 - 3\sqrt{(-5)})},$$

we have

$$3 = \alpha\alpha', 7 = \beta\beta', 1 + 2\sqrt{(-5)} = -\alpha'\beta, 1 - 2\sqrt{(-5)} = -\alpha\beta',$$

$$3 \cdot 7 = \alpha\alpha' \cdot \beta\beta' = \alpha'\beta \cdot \alpha\beta' = (1 + 2\sqrt{(-5)})(1 - 2\sqrt{(-5)}) ;$$

and all of these equations are entirely natural. *In order to obtain a satisfactory theorem of factorization in K^*, we must conceive K^* as immersed in the larger field L.* The logic of the solution is exactly the same as that of the solution of the corresponding, but trivial, problem for the numbers $4m+1$.

On the other hand there is an obvious contrast between the two solutions. It is *natural* to think of the field "$4m+1$" as part of the field "m"; "m" is the more obvious and simpler field. It is not natural to think of K^* as part of L; K^* is a much simpler and more natural field than L, and we should like to do without the reference to the latter if we could. It will be very tiresome if, whenever we consider an algebraic field, we are to be compelled to construct some more elaborate field of which it is a part. We should prefer to tidy up the house without going out of doors.

We may look for a hint once more in the numbers $4m+1$. Some of these numbers are divisible by 7, a number outside the field; and these numbers stand in certain specific relations to one another inside the field. Could we give a rational account of these relations without explicit reference to the number 7? It is a very unnatural thing to try to do, since what is *important* about the numbers is precisely that they *are* divisible by 7, but we could do it; we could define the class

$$21, 49, 77, 105, \cdots,$$

of numbers $4m+1$ divisible by 7 in terms of the field $4m+1$ itself. For example, we could take the first two numbers 21 and 49, and say "the class in question is the class which begins with these two numbers and whose members recur at regular intervals in the field." It is of course an artificial definition, and it is impossible to conceal from ourselves what we are really doing.

It is often a very profitable exercise for a mathematician to force himself to solve some simple problem without the weapon obviously appropriate to the occasion, to throw away the key of the front door and insist on forcing himself in somehow through the window. The forced and unnatural solution of one problem will often turn out to contain the germ of a quite natural solution of another. So it proves in this case; it is natural to try to define the numbers of K^* divisible by ξ without going outside K^*; it is natural, and possible, and it gives us the key to what is, in the general case, the established method of constructing a satisfactory arithmetic.

It is obvious that, if α and β belong to K^*, and $\xi \mid \alpha$ and $\xi \mid \beta$, then $\xi \mid \lambda\alpha+\mu\beta$, where λ and μ are any numbers of K^*. The converse proposition is not true; it is not true that if I is any set of numbers of the field K^* which has the property "if α and β belong to I, then $\lambda\alpha+\mu\beta$ belongs to I, for every λ and μ of the field", then there exists a number ξ, belonging to K^* which divides every number of I. What *is* true is that every number of I is divisible by a ξ which belongs to L but not in general to K^*. The set I is identical with the set of numbers of K^* divisible by ξ. Such a set I, or the more general set based on any finite number of numbers $\alpha, \beta, \gamma, \cdots$, of K^*, is called an *ideal*, the numbers ξ, underlying K^* but not belonging to it, having been described by Kummer as "ideal numbers". In ordinary arithmetic ideals are simply the sets of numbers divisible by some special number such as 3, and there is nothing in particular to be gained by their introduction. In an algebraic field they are not, in general, the sets of numbers divisible by a number *of the field*, and their introduction is essential before arithmetic can get properly started. We can define multiplication and division of ideals, prime ideals, and so on, and when we have done this we find that the arithmetic of ideals has all the properties of ordinary multiplicative arithmetic. In particular, *every ideal can be resolved uniquely into prime ideals*; the fundamental theorem is true when stated in terms of ideals.

The proof of the fundamental theorem is not particularly difficult; Landau presents it, with all the preliminary definitions, in about a dozen pages of quite simple reasoning. But I would not commit the impertinence, even if I had the time, of assuming the airs of an expert in the algebraic theory of numbers, a subject which I admire only at a distance and in which I have never worked. It is ordinary rational arithmetic which attracts the ordinary man, and I have digressed outside it only because there is a good deal in it which it is impossible to appreciate properly without a little knowledge of the larger theory. It is impossible, for example, to appreciate Euclid's arithmetical achievements until we realize that there are arithmetics in which the most obvious analogs of his theorems are false.

15. *Conclusion. Pedagogy.* There are few things in the world for which I have less taste than I have for mathematical pedagogics, but I cannot resist the temptation of concluding with one pedagogic lesson. There was, and I fear still is, a popular English text book of algebra which I used at school and which contained a chapter on the theory of numbers. It might be expected that such a chapter would be among the most instructive in the book; we might suppose, for example, that Euclid's algorithm, with its elegance, its simplicity, and its far reaching consequences, would be an ideal text for the instruction of a bright young mathematician. In fact the algorithm was never mentioned; one was to find the highest common factor of 12091 and 14803, I suppose, by "trial"; and all that the authors had to say of the fundamental theorem was that "it is so evident that it may be regarded as a necessary law of thought." It is possible of course that all this may have been expunged from later editions. It is certain, however, that chapters on number theory in textbooks of algebra are usually quite intolerably bad, and it is conceivable that Oxford University may have been right in erasing the subject altogether from its more elementary examination schedule.

The elementary theory of numbers should be one of the very best subjects for early mathematical instruction. It demands very little previous knowledge; its subject matter is tangible and familiar; the processes of reasoning which it employs are simple, general and few; and it is unique among the mathematical sciences in its appeal to natural human curiosity. A month's intelligent instruction in the theory of numbers ought to be twice as instructive, twice as useful, and at least ten times as entertaining as the same amount of "calculus for engineers". It is after all only a minority of us who are going to spend our lives in engineering workshops, and there is no particular reason why most of us should feel any overpowering interest in machines; nor is it in the least likely that, on those occasions when machines are of real importance to us, we shall require the power of dealing with them by methods more elaborate than the simplest rule of thumb. It is not engineering mathematics that is wanted for the understanding of modern physics, and still less is it wanted by most of us for the ordinary needs of life; we do not actually drive cars by solving differential equations. There may be a case for subordinating mathematics to the linguistic and literary studies which are so much more obviously useful to ordinary men, but there is none for sacrificing a splendid subject to meet a quite imaginary need.

PRINCETON UNIVERSITY

NOTE ON A CERTAIN TYPE OF PARABOLA*

BY W. C. GRAUSTEIN

In the development of the geometry of the complex domain, there has been a tendency to pass over problems of a relatively elementary character, despite their central position and importance. It is the purpose of this note to call attention to one of these problems, namely that of the classification of complex conics with respect to the group of complex rigid motions. A systematic study of this problem brings to light two *special* types of non-degenerate conics: the non-degenerate central conics which contain just one circular point at infinity, and the non-degenerate parabolas which are tangent to the line at infinity at a circular point. These special conics have already been considered, from a different point of view, in this Bulletin.†

It is not our intention to go into detail, either in connection with the general problem of classification or in consideration of the special conics. We propose merely to discuss certain particularly striking properties of the special parabolas.

It is evident that the complex conic

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \tag{1}$$

A, B, C not all zero, is a special parabola if and only if

$$B^2 - 4AC = 0, \quad A + C = 0, \quad \Delta \neq 0, \tag{2}$$

where Δ is the discriminant of (1).

Since a special parabola has neither focus nor directrix, the question of its eccentricity cannot be handled directly. We may, however, establish an equation determining the

* Presented to the Society, April 6, 1928.

† E. V. Huntington and J. K. Whittemore, *Some curious properties of conics touching the line at infinity at one of the circular points*, vol. 8 (1901–02), pp. 122–124.

eccentricities of the general conic (1) and extend its validity to all conics. An equation with this property, whose roots are the *squares* of the eccentricities, is

$$(B^2 - 4AC)s^2 - 4[(A + C)^2 + B^2 - 4AC](s - 1) = 0.$$

For a special parabola the coefficients in this equation are all zero. Hence the homogeneous functions of degree zero in A, B, C, which for the general conic are the squares of the eccentricities, are, for a special parabola, undefined.

THEOREM 1. *A special parabola has no eccentricity* and is the only conic with this property.*

Consider a special parabola tangent to the line at infinity at $J:(1,-i,0)$. Let the parabola be referred to the point of contact, O, of the isotropic tangent through $I:(1,i,0)$ as origin of a system of rectangular Cartesian coordinates. Then regardless of the choice of axes of coordinates, the equation of the parabola takes the form

$$(x - iy)^2 = 2k(x + iy), \qquad k \neq 0,$$

or, in terms of isotropic coordinates, the form

$$\bar{z}^2 = 2kz, \qquad k \neq 0. \tag{3}$$

The rotation about O through the angle θ:

$$z' = e^{\theta i}z, \quad \bar{z}' = e^{-\theta i}\bar{z}, \tag{4}$$

carries the parabola (3) into $\bar{z}'^2 = 2ke^{-3\theta i}z'$ and hence, when $3\theta \equiv 0 \pmod{2\pi}$, into itself. Herewith, we have the most bizarre of peculiarities imaginable for a conic.

THEOREM 2. *A special parabola has rotational symmetry about the point of contact O of its unique isotropic tangent: there exists a group of three rotations about O which carry it*

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* It is evident that a variable conic may be made to approach a special parabola as a limit in such a way that the square of one of its eccentricities approaches any preassigned value, 0 *not* excluded. Herein we have, for example, the explanation of the presence of two special parabolas in a pencil of equilateral hyperbolas. See Huntington and Whittemore, loc. cit.

into itself, namely the rotations through 0, $2\pi/3$, *and* $4\pi/3$ *radians.*

It follows that there is an infinity of equilateral triangles inscribed in the parabola, all of which have the point O as median or central point. Every point of the parabola, except O and the point at infinity, is a vertex of just one of the triangles.

By means of equations (4), interpreted as a rotation of the axes of coordinates, equation (3) of our special parabola may be reduced to the normal form

$$\bar{z}^2 = 2z. \tag{5}$$

It suffices to choose θ so that $e^{3\theta i} = k$. Hence:

THEOREM 3. *There are only two types of special parabolas: every two special parabolas with the same point at infinity are equivalent with respect to the group of complex rigid motions.*

Here, again, the special parabolas differ from the ordinary ones. The ordinary nondegenerate parabolas tangent to the line at infinity at a given (noncircular) point are of an infinity of types depending on one complex parameter. The parameter may be taken as the square of the semi latus rectum, whose value for an ordinary nondegenerate parabola (1) is given by an expression, $\Delta/(A+C)^3$, which is undefined (infinite) for every special parabola.

An instructive side-light on Theorem 2 is obtained by studying its projective generalization. Consider a complex collineation of general type—of which a rotation is a special metric case—and write its equations in the canonical forms

$$x_1' = a_1 x_1, \qquad x_2' = a_2 x_2, \qquad x_3' = a_3 x_3, \tag{6}$$

where a_1, a_2, a_3 are complex numbers, no two equal and no one zero. If the collineation leaves fixed a nondegenerate conic, it establishes on the conic a projectivity which is readily shown to be nonparabolic. The conic must, therefore, be tangent to two fixed lines of the collineation at two fixed

points; that is, belong to one of the three pencils of conics

$$(7) \quad k_1x_1^2 = 2l_1x_2x_3, \quad k_2x_2^2 = 2l_2x_3x_1, \quad k_3x_3^2 = 2l_3x_1x_2.$$

The degenerate conics of these pencils are obviously invariant with respect to the collineation. Hence, if one of the nondegenerate conics of a pencil is invariant, all the conics of the pencil are invariant. The conditions for invariance in the three cases are respectively

$$a_1^2 = a_2a_3, \qquad a_2^2 = a_3a_1, \qquad a_3^2 = a_1a_2.$$

Any one of these relations is a consequence of the other two. The collineations (6) determined by them are

$$(8) \quad \begin{aligned} &x_1' = \omega^2x_1, \ x_2' = \omega x_2, \ x_3' = x_3 \ ; \\ &x_1' = \omega x_1, \ \ x_2' = \omega^2x_2, \ x_3' = x_3, \end{aligned}$$

where ω is a primitive cube root of unity.

THEOREM 4. *The only collineations of general type* (6) *which leave fixed each of the conics of two, and hence of all three, of the pencils of conics* (7) *are the collineations* (8).

If two of the three fixed points of the collineation (6) are the circular points at infinity, two of the pencils (7) are those of the special parabolas having the odd fixed point as the point of contact of their unique isotropic tangents, the third pencil is that of the circles with the odd fixed point as center, and the two collineations (8) are the rotations about the odd fixed point through $2\pi/3$ and $4\pi/3$ radians.

One last remark. A real nondegenerate conic (1) is a circle if and only if

$$(A + C)^2 + B^2 - 4AC = 0 \text{ or } (A - C)^2 + B^2 = 0.$$

But the complex nondegenerate conics (1) defined by this relation include, besides the circles, the special parabolas and the special central conics. Moreover, both eccentricities of a special central conic, as well as both those of a circle, are zero.

HARVARD UNIVERSITY

ON ORDINARY RESTRICTED EXTREMA IN CONNECTION WITH POINT TRANSFORMATIONS

BY JACQUES HADAMARD

1. *Restricted Minima in a Plane.* The question which I shall examine concerns the maximum or minimum of $f(x, y)$, when the variables x and y are not independent, but subjected to the condition

$$(g) \qquad g(x, y) = 0,$$

so that the point (x, y) must describe the curve represented by the preceding equation.

This is what was called, formerly, a *relative* extremum, but what I proposed to call a *restricted** extremum, because the term "relative" is used with another meaning. Indeed, the extrema which we consider, as every extremum treated by the methods of differential calculus, are *relative*, that is, they are extrema only in comparison with neighboring values of f.

The first part of the question, namely, the investigation of stationary values of f on the curve (g), is quite classic. It is usually solved, with the help of the Lagrange multiplier l, by means of the simultaneous equations

$$(1) \qquad p + lp_1 = 0, \qquad q + lq_1 = 0,$$

where p and q are the partial derivatives of f, and p_1 and q_1 are those of g. However, we shall use here the condition with l eliminated, that is,

$$(j) \qquad j = pq_1 - qp_1 = 0,$$

* The author informs the editors that he, together with David Hilbert and others, had previously agreed that a change of nomenclature is desirable, to replace the older term relative extrema. The agreement is to use "*extremum libre ou lié*" in French, and "*freies oder gebundenes Extremum*" in German. The author had used "*bound*" in English, and the editors have suggested the word "*restricted*." THE EDITORS.

so that j is the jacobian of f and g. A point m (x_0, y_0) of (g) at which f is stationary on (g) necessarily belongs to the curve defined by (j). We shall suppose, throughout this paper, that m is an ordinary point of (g) and of (j). At m, the curve $f(x, y) = f(x_0, y_0)$ is tangent to (g).

This settles the question of stationary values. But there remains the question whether such a value, when once found, is a maximum or a minimum. This, of course, can be answered easily if we begin by imagining that x and y, the coordinates of an arbitrary point of (g), are expressed in terms of a variable parameter u. A maximum or a minimum will occur, according to the sign of d^2f/du^2. Then, expanding the latter derivative and taking account of the fact that

$$dg = p_1 dx + q_1 dy = 0, \quad d^2g = 0,$$

since the point (x, y) must describe the curve (g), it is easily found that the sign involved is that of

$$(2) \qquad (r + lr_1)q_1^2 - 2(s + ls_1)p_1q_1 + (t + lt_1)p_1^2,$$

where r, s, t denote second derivatives of f, and r_1, s_1, t_1 denote second derivatives of g.

But it is worth while to notice that an equivalent, completely different solution may be obtained by the study of the point transformation

$$(T) \qquad X = f(x, y), \qquad Y = g(x, y),$$

in the neighborhood of m.

At m itself, the jacobian j of that transformation vanishes, and the curve (j) divides the plane (or, more exactly, the region surrounding m) into two parts σ and σ', the first of which will correspond to $j > 0$, the second to $j < 0$. Now, we know* that, in the XY plane, these two regions will have images located on *one and the same side* of the line (J) which corresponds to (j): we can say that, in the vicinity of the point M, which is the image of m, the images of σ and σ' only give one area R, limited by (J).

* See my *Cours d'Analyse*, vol. I.

In order to solve the given question, we need only inquire whether a direction issuing from M and having α, β for its directing cosines, is directed towards the inside of R.

Let us call *direction parameters* of a direction, any two quantities proportional to the direction cosines of that direction, *with a positive coefficient of proportionality*. In particular, the normal to (j) at m, directed towards the inside of σ, will admit of direction parameters equal to the partial derivatives of j,

$$P = \frac{\partial j}{\partial x}, \quad Q = \frac{\partial j}{\partial y}.$$

Then let us consider the direction tangent to (j), directed with respect to σ. It will admit of the direction parameters Q, $-P$, so that an infinitesimal displacement (dx, dy) of m on (g) in the direct sense will satisfy the relations

$$\frac{dx}{Q} = \frac{dy}{-P} > 0.$$

Hence the corresponding displacement in the XY plane will be such that

$$\frac{dx}{pQ - qP} = \frac{dy}{p_1Q - q_1P} > 0. \tag{3}$$

The latter displacement will be tangent to (J), in the direct sense with respect to R.

Therefore a necessary and sufficient condition that a direction issuing from M and not tangent to (J) be directed towards the inside of R, is that it shall determine, with the direction defined by (3), an angle with a positive sine. If α and β denote the direction cosines of such a direction, this condition may be written in the form

$$\beta(pQ - qP) - \alpha(p_1Q - q_1P) > 0. \tag{4}$$

The application to restricted maxima or minima of f on the line (g) is obvious. Let us consider an arc of the latter line, which shall contain m and be sufficiently small. Its image in

the XY plane will be a portion of a parallel to the X axis, *which must lie inside R.* If we take its origin at M, its direction cosines, which are $+1$ and 0, must therefore, satisfy the condition (4). We see, then, that the proper sign is plus if we have

$$p_1Q - q_1P < 0, \tag{5}$$

and minus if

$$p_1Q - q_1P > 0. \tag{5'}$$

In the first case, X_0 negative, the value of f at m, which is the abscissa of M, will be a restricted *minimum* of f under the given conditions; in the second case, a *maximum.*

Similarly, the curve $f(x, y) = f(x_0, y_0)$, which goes through m, will have for its image a portion of the straight line $X = X_0$, and the direction from M thus defined will be parallel to the positive Y axis or to the negative Y axis, according to the sign of the quantity

$$pQ - qP.$$

This quantity and the quantity $p_1Q - q_1P$ have the ratio $-l$, on account of (1). The fact that a restricted maximum of f when g is constant coincides with a maximum or with a minimum of g when f is constant, according to the sign of l, is classic.

It is easy to verify by direct computation that the condition thus obtained is the same as that which would result from the expression (2). The remarkable thing is that the latter can be expressed in terms of the *first* derivatives of f, g, and j.

2. *The Case of a Curve in Space.* Let us consider the curves defined by the equations

$$g_1(x, y, z) = 0, \quad g_2(x, y, z) = 0,$$

and a stationary value, let us say at the (ordinary) point $m(x_0, y_0, z_0)$, of $f(x, y, z)$ on that curve.

At m, the jacobian

$$j = \begin{vmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \\ \dfrac{\partial g_1}{\partial x} & \dfrac{\partial g_1}{\partial y} & \dfrac{\partial g_1}{\partial z} \\ \dfrac{\partial g_2}{\partial x} & \dfrac{\partial g_2}{\partial y} & \dfrac{\partial g_2}{\partial z} \end{vmatrix} = \begin{vmatrix} p & q & r \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix}$$

must vanish. Again, we shall suppose that m is an ordinary point of the surface $j=0$, so that not all the three derivatives

$$P = \frac{\partial j}{\partial x}, \quad Q = \frac{\partial j}{\partial y}, \quad R = \frac{\partial j}{\partial z}$$

are zero. Such a surface (j) will divide the space near m into two regions ω, ω', the first of which will correspond to $j>0$. Let us consider the point transformation defined by

$$X = f(x,y,z), \quad Y = g_1, \quad Z = g_2.$$

Around M (the image of m), ω and ω' will have a common image R, on one side of the surface (J) which corresponds to j. Let us find a necessary and sufficient condition that the direction from M with the direction cosines α, β, γ be directed towards the inside of R. Let (dx, dy, dz), $(\delta x, \delta y, \delta z)$ be two directions from m, tangent to (j), so that

$$(6) \qquad \frac{dy\,\delta z - dz\,\delta y}{P} = \frac{dz\,\delta x - dx\,\delta z}{Q} = \frac{dx\,\delta y - dy\,\delta x}{R}.$$

These two directions, in the above order, will define a direct orientation on s, at m, if the common value of the above ratios is positive. In the XYZ space, the corresponding directions will be

$$(7) \qquad \begin{aligned} dX &= p dx + q dy + r dz, \quad dY = p_1 dx + q_1 dy + r_1 dz, \\ dZ &= p_2 dx + q_2 dy + r_2 dz, \\ \delta X &= p\delta x + q\delta y + r\delta z, \quad \delta Y = p_1\delta x + q_1\delta y + r_1\delta z, \\ \delta Z &= p_2\delta x + q_2\delta y + r_2\delta z, \end{aligned}$$

so that the tangent line to s at M will have direction cosines A, B, C, such that

$$\frac{A}{dY\delta Z - dZ\delta Y} = \frac{B}{dZ\delta X - dX\delta Z} = \frac{C}{dX\delta Y - dY\delta X}.$$

Now, our direction (α, β, γ) will be directed towards R if it gives with (7) a direct trihedron, that is, if

$$\begin{vmatrix} dX & dY & dZ \\ \delta X & \delta Y & \delta Z \\ \alpha & \beta & \gamma \end{vmatrix} > 0.$$

Such will be the case for a parallel to the positive X axis if

$$dY\delta Z - dZ\delta Y > 0,$$

or, on account of (6) and (7),

$$P(q_1r_2 - r_1q_2) + Q(r_1p_2 - p_1r_2) + R(p_1q_2 - q_1p_2) > 0.$$

An analogous solution for extrema on a surface would be interesting to find.

3. *Conclusion.* Let us come back to the case of the plane. The above argument rests on the properties of the transformation at a point where its jacobian vanishes, which is especially interesting to study from the geometric point of view, as will be done in a paper by Professor Julia to appear shortly in the Bulletin des Sciences Mathématiques. The fundamental fact is that, at such a point m, any two curves of the xy plane will, in general, have images which will be tangent to each other; but there is an exception, concerning a certain peculiar direction at m. Any curves tangent to that special direction will have ordinarily in the XY plane, an image that has a cusp, or sometimes a terminal point.* The latter case (a particular case of the former) occurs precisely for our curves $f(x, y) = f(x_0, y_0)$, $g(x, y) = 0$.

COLLÈGE DE FRANCE

* The words *terminal point* are suggested by the editors as a translation of the French term *point d'arrêt*.

A CERTAIN BIRATIONAL TRANSFORMATION OF ORDER r^2 BETWEEN TWO r-SPACES IN AN $(r+1)$-SPACE

BY B. C. WONG

In this paper we present a birational transformation of order r^2 between two r-spaces S_r and S_r' in an $(r+1)$-space S_{r+1}. If $r=2$, we have the well known quartic birational transformation between two planes of an S_3 obtained by intersecting with the two planes the bisecants of a twisted cubic curve in S_3. A point of one plane and a point of the other are said to be corresponding points if they lie on the same bisecant of the cubic curve. It is our object to generalize this construction to hyperspace.

For this purpose we let two r-ic hypersurfaces (necessarily ruled) intersect in a composite manifold composed of an M_{r-1}^{n} of order $n=r(r+1)/2$ and an $M_{r-1}^{n'}$ of order $n'=r(r-1)/2$. The former component manifold, M_{r-1}^{n}, will have just one apparent r-fold point* if it meets the latter, M_{r-1}, in an $(r-2)$-dimensional variety of order $r(r^2-1)/3$. This implies that M_{r-1}^{n} and $M_{r-1}^{n'}$ are such that a general S_3 meets them in two curves C^n and $C^{n'}$, respectively, having $r(r^2-1)/3$ points in common and that C^n has $r(r^2-1)\cdot(3r-2)/24$ and $C^{n'}$ has $r(r-1)(r-2)(3r-5)/24$ apparent double points. It also implies that a general S_4 meets them in two surfaces having respectively $r(r+1)(r-1)^2(r-2)^2/48$ and $r(r-1)(r-2)^2(r-3)^2/48$ apparent triple points. The number of apparent $(k-1)$-fold points on their intersections with an S_k can also be found.†

As we are going to make use of M_{r-1}^{n}‡ to obtain an r^2-ic

* By an apparent r-fold point of a V_{r-1} in S_{r-1} we mean a line passing through a general point of S_{r+1} and meeting V_{r-1} r times.

† See B. C. Wong, *On the number of apparent triple points of surfaces in space of four dimensions*, this Bulletin, vol. 35, No. 3, pp. 339–343.

‡ There are in S_{r+1} many $(r-1)$-dimensional varieties of order m $[2r-1\leqq m\leqq r(r+1)/2]$ that have just one apparent r-fold point but the M_{r-1}^{n} here described is the only one that offers the desired transformation.

birational transformation between S_r and S_r', we make a few remarks concerning the manifold. It can be shown that this manifold can be represented upon an S_{r-1} by means of an ∞^{r+1}-system of $(r+1)$-ic $(r-2)$-dimensional varieties passing through an $(r-3)$-dimensional variety of order $(r+1)(r+2)/2$ in S_{r-1}. The case $r=3$ is well known: the points of M_2^6 are in a one-to-one correspondence with the points of a plane, the fundamental curves of representation being quartic curves through 10 points in the plane.

M_{r-1}^n is also the locus of points whose polar r-spaces with respect to r hyperquadrics $Q_r^{(i)}$ $[i=1, 2, \cdots, r]$ in S_{r+1} meet in lines all lying in an S_r. The polar r-spaces of points of S_r with respect to $Q_r^{(i)}$ meet in lines r-uply secant to M_{r-1}^n. Again, an S_r meets M_{r-1}^n in a V_{r-2}^n which is the Jacobian variety of the r quadric varieties in which S_r meets $Q_r^{(i)}$. The polar $(r-1)$-spaces of the points of V_{r-2}^n with respect to the same quadric varieties of S_r meet in lines intersecting V_{r-2}^n r times and forming a $V_{r-1}^{r^2-1}$ of order r^2-1 on which V_{r-2}^n lies r-uply.

If we replace r by $r-1$ in n, the M_{r-1}^n of S_{r+1} becomes an $M_{r-2}^{n'}$ of S_r which is of the same nature as the variety in which S_r meets $M_{r-1}^{n'}$. Now $M_{r-1}^{n'}$ is such that a general S_{r-1}-section of it is the Jacobian variety of $r-1$ $(r-2)$-dimensional quadric varieties in S_{r-1}. It is to be noticed that $M_{r-1}^{n'}$ is a ruled manifold composed of ∞^{r-2} lines all meeting M_{r-1}^n r times. These lines or the points of the variety in which an S_r meets $M_{r-1}^{n'}$ can be set in a one-to-one correspondence with the points of an S_{r-2}.

Now let two r-spaces S_r and S_r' be given in S_{r+1} and let S_r meet M_{r-1}^n in a V_{r-2}^n and S_r' meet M_{r-1}^n in a $V_{r-2}^{'n}$. These two varieties, V_{r-2}^n and $V_{r-2}^{'n}$, have in common a V_{r-3}^n which lies in the R_{r-1} of intersection of S_r and S_r'. The r-fold secants of V_{r-2}^n form a $V_{r-1}^{r^2-1}$ and those of $V_{r-2}^{'n}$ form a $V_{r-1}^{'r^2-1}$. These two r-fold secant varieties both intersect R_{r-1} in the same $V_{r-2}^{r^2-1}$ which contains V_{r-3}^n r-uply.

Since M_{r-1}^n has just one apparent r-fold point, from a point P of S_r we draw the r-fold secant to M_{r-1}^n meeting S_r' in

a point P' which is said to correspond to P. As the construction is reversible, the correspondence is birational. To show that the correspondence is of order r^2, that is, to an $(r-1)$-space of one of the given r-spaces corresponds a $V_{r-1}^{r^2}$ of order r^2 of the other, we notice that the ∞^{r-1} r-uple secant lines of M_{r-1}^{n} that meet an S_{r-1} form a $V_r^{r^2}$, for an S_r containing S_{r-1} meets it in a composite variety composed of S_{r-1} and a $V_{r-1}^{r^2-2}$. Hence if P describes an S_{r-1} in S_r, the corresponding r-fold secant of M_{r-1}^{n} describes a $V_r^{r^2}$ which is met by S_r' in a $V_{r-1}'^{r^2}$. Similarly, there is a $V_{r-1}^{r^2}$ in S_r corresponding to an S_{r-1}' in S_r'.

Now $V_{r-1}^{r^2}$ passes through the variety V_{r-2}^{n} of intersection of M_{r-1}^{n} with S_r, r times; and through the $V_{r-2}^{r^2-1}$ in R_{r-1}, once. Since, from the very nature of the transformation, every point of R_{r-1} not on $V_{r-2}^{r^2-2}$ is its own image, $V_{r-1}^{r^2}$ has in common with R_{r-1}, besides the $V_{r-2}^{r^2}$, an S_{r-2} in which R_{r-1} meets the S_{r-1}' to which $V_{r-1}^{r^2}$ corresponds.

Now to show that to a line in one r-space corresponds an r^2-ic curve in the other. Let P describe a line l in S_r. The line l meets the variety $V_{r-1}^{r^2-1}$ of lines r-uply secant to the V_{r-2}^{n} in which S_r meets M_{r-1}^{n} in r^2-1 points, i.e., meets r^2-1 of the lines of $V_{r-1}^{r^2-1}$. Hence the ∞^1 r-fold secants of M_{r-1}^{n} that meet l form a ruled surface F^{r^2} of order r^2, for it is met by S_r in a composite curve made up of r^2 lines one of which is l. Now F^{r^2} is met by S_r' in a curve C'^{r^2} which is the image of l in S_r.

Similarly, it can be shown that, if a point describes a k-space in the one r-space, the corresponding point in the other r-space describes a k-dimensional variety of order $\binom{r}{k}^2$.

If we project S_r' upon S_r, we have an involutorial transformation of order r^2 in S_r. Attention is here called to the fact that this involutorial correspondence is the product of two involutorial r-ic correspondences. One such can be set up by means of r of the $r(r-1)$-dimensional quadric varieties $Q_{r-1}^{(i)}$ in S_r. To a point P we make correspond the point P' of intersection of the r polar $(r-1)$-spaces with respect to $Q_{r-1}^{(i)}$. If P describes an S_k, P' describes a V_k^{μ}, where $\mu=\binom{r}{k}$. By setting up in S_r another involutorial r-ic transformation in

the same manner, we obtain as the image of V_k^n a $V_k^{n^2}$ which is of the same nature as that obtained by means of the r^2-ic transformation as the image of an S_{r-1}.

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ON SOME FUNCTIONS CONNECTED WITH $\phi(n)$

BY S. SIVASANKARANARAYANA PILLAI*

Let $\phi(n)$ denote, as usual, the number of numbers not greater than and prime to n. Let $N(x)$ be the number of distinct numbers less than x, which can be the ϕ function of some number; and let $R(n)$ be the number of solutions of the equation

$$n = \phi(x),$$

n being given. The object of this note is to prove some results concerning the magnitude of $N(n)$ and to apply them to prove that

$$\overline{\lim_{n=\infty}} R(n) = \infty .$$

Since there is no reference to such results in Dickson's *History of the Theory of Numbers*, I believe that the last result in particular is new.

THEOREM I. *We have*

$$N(n) > \frac{a \cdot n}{\log n},$$

where a is a constant.

PROOF. For each prime p, $\phi(p) = p - 1$; hence, if we denote by $\pi(n)$ the number of primes not exceeding n, then

$$N(n) \geqq \pi(n).$$

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* I take this opportunity to express my deep gratitude to K. Ananda Rao for his invaluable guidance and encouragement.

This paper was read before the conference of the Indian Mathematical Society held in December 1928.

But by elementary methods it has been proved that*

$$\pi(n) > \frac{a \cdot n}{\log n}$$

where $a>0$ is a constant. Therefore

$$N(n) > \frac{a \cdot n}{\log n} \cdot$$

THEOREM II. *We have*

$$N(n) = O\left\{\frac{n}{(\log n)^t}\right\},$$

where $t=(\log 2)/e$.

PROOF. If p is an odd prime, $\phi(p^\alpha)$ is even; and $\phi(m \cdot n) = \phi(m)\phi(n)$ when m and n are prime to each other. Therefore, if any number m is composed of more than r different odd prime factors, then $\phi(m)$ is divisible by 2^{r+1}. So, if a number of the form $2^r \cdot h$ (where h is odd) should be a $\phi(m)$, then m may contain at most r different odd prime factors. Consequently, in the set $2^s \cdot h_s$, where s takes the values $0, 1, 2, \cdots, r$, and h_s runs through all odd numbers not exceeding $n/2^s$, the number of numbers which can be the ϕ of some numbers, is not greater than

$$\pi_1(n) + \pi_2(n) + \cdots + \pi_{r+1}(n),$$

where $\pi_r(x)$ is the number of numbers not exceeding x, which are composed of r different prime factors. But the number of numbers in the set $2^s \cdot h_s$ considered above, is

$$= \sum_{0 \leqq s \leqq r} \left[\frac{n}{2^{s+1}}\right] = \sum_{0 \leqq s \leqq r} \frac{n}{2^{s+1}} + O(r) = n - \frac{n}{2^{r+1}} + O(r).$$

Hence, of the numbers $\leqq n$, at least

$$n - \frac{n}{2^{r+1}} + O(r) - \sum_{1 \leqq s \leqq r+1} \pi_s(n)$$

numbers cannot be the ϕ function of any number. Therefore

* Ramanujan's *Collected Papers*, pp. 208–209. Landau, *Vorlesungen über Zahlentheorie*, vol. 1; Theorem 112. etc.

$$N(n) \leqq n - \left\{ n - \frac{n}{2^{r+1}} + O(r) - \sum_{1 \leqq s \leqq r+1} \pi_s(n) \right\}$$

$$= \frac{n}{2^{r+1}} + O(r) + \sum_{1 \leqq s \leqq r+1} \pi_s(n).$$

By elementary methods, Hardy and Ramanujan have proved that*

$$\pi^{\gamma}(n) < \frac{kn(\log\log n + c)^{\gamma-1}}{(\gamma - 1)! \log n},$$

where k and c are constants.

Hence, if $r+1 < \log\log n$,

$$\sum_{1 \leqq s \leqq r+1} \pi_s(n) = O\left(\sum_{1 \leqq s \leqq r+1} \frac{n(\log\log n + c)^{s-1}}{(s-1)! \log n} \right)$$

$$= O\left(\sum_{1 \leqq s \leqq r+1} \frac{n(\log\log n)^{s-1}}{(s-1)! \log n} \right),$$

for

$$\left(1 + \frac{c}{\log\log n}\right)^{s-1} \leqq \left(1 + \frac{c}{\log\log n}\right)^{\log\log n}$$

$$= e^{\log\log n \log(1 + c/\log\log n)} \leqq e^{c} = O(1).$$

But, since $r+1 < \log\log n$,

$$\frac{(\log\log n)^{s-1}}{(s-1)!} < \frac{(\log\log n)^{s}}{s!}.$$

Therefore, if $r+1 < \log\log n$,

$$\sum_{1 \leqq s \leqq r+1} \pi_s(n) = O\left(\frac{r \cdot n}{\log n} \frac{(\log\log n)^{r}}{r!} \right).$$

Therefore, if $r+1 < \log\log n$,

$$N(n) = O\left(\frac{n}{2^{r+1}}\right) + O(r) + O\left(\frac{n}{\log n} \frac{(\log\log n)^{r}}{(r-1)!} \right)$$

$$= O(s_1) + O(s_2) + O(s_3), \text{ say.}$$

Put

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* Ramanujan's *Collected Papers*. Paper No. 35, 2.2, Lemma A.

$$r = \left[\frac{\log\log n}{e}\right].$$

Then, by Stirling's theorem,

$$\begin{aligned}
\log s_2 &= \log n - \log\log n + r\log\log\log n - (r-1)\log(r-1) \\
&\quad - \frac{1}{2}\log(r-1) + r - 1 + O(1) \\
&\leqq \log\log n - \log\log n + \frac{\log\log n \log\log\log n}{e} \\
&\quad - \frac{\log\log n}{e}\log\frac{\log\log n}{e} - \frac{1}{2}\log\frac{\log\log n}{e} \\
&\quad + \frac{\log\log n}{e} + O(1) \\
&= \log n - \log\log n\left(1 - \frac{1}{e} - \frac{1}{e}\right) - \frac{1}{2}\log\log\log n \\
&\quad + O(1) \leqq \log n - \log\log n\left(\frac{e-2}{e}\right) + O(1) \\
&\leqq \log n - \frac{\log 2}{e}\log\log n + O(1) \\
&= \log n - t\log\log n + O(1).
\end{aligned}$$

Therefore

$$s_3 = O\left(\frac{n}{(\log n)^t}\right),$$

$$s_2 = O(r) = O(\log\log n) = O\left(\frac{n}{\log^t n}\right),$$

$$s_1 = O\left(\frac{n}{2^{\log\log n/e}}\right) = O\left(\frac{n}{\log^t n}\right).$$

Therefore

$$\begin{aligned}
N(n) &= O(s_1 + s_2 + s_3) \\
&= O\left(\frac{n}{\log^t n}\right).
\end{aligned}$$

Now we shall apply the above result to prove the following theorem.

THEOREM III. *We have*

$$R(n) \neq o(\log^t n)$$

and in particular,

$$\overline{\lim_{n=\infty}} R(n) = \infty .$$

PROOF. Let

$$s(n) = \sum_{1 \leqq m \leqq n} R(n) .$$

If possible, let

$$R(m) = o(\log m)^t .$$

Then

$$\begin{aligned} s(n) &\leqq N(n)\Big(\max_{1 \leqq m \leqq n} R(m)\Big) \\ &= N(n)\{o(\log^t n)\} \\ &= o\left(\frac{n}{\log^t n}\right)\{O(\log^t n)\} \\ &= o(n) , \end{aligned}$$

from Theorem II.

Now, $S(n)$ is the number of numbers whose ϕ functions are $\leqq n$. But, since $\phi(m) < m$, $S(n) \geqq n$, which contradicts $S(n) = o(n)$. The theorem is therefore proved.

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ON A FUNCTION CONNECTED WITH $\phi(n)$

BY S. SIVASANKARANARAYANA PILLAI

Let $\phi(n)$ be the number of numbers not greater than and prime to n. Further, let

$$\phi_1(n) = \phi(n),$$
$$\phi_2(n) = \phi\{\phi_1(n)\},$$
$$\phi_3(n) = \phi\{\phi_2(n)\},$$
$$\cdots\cdots\cdots$$
$$\phi_{r+1}(n) = \phi\{\phi_r(n)\}.$$

It is obvious that at one stage

$$\phi_r(1) = 1.$$

Hence, with an integer n, there is associated another $R(n)=r$, such that r is the least integer for which

$$\phi_r(n) = 1.$$

This short note is about this function $R(n)$.*

THEOREM I.

$$R(n) \leqq \left[\frac{\log n}{\log 2}\right] + 1.$$

PROOF. When $n>2$, $\phi(n)$ is always even. If x and y are even, and y contains at least one odd prime factor and x does not, then

$$\frac{\phi(x)}{x} > \frac{\phi(y)}{y}.$$

Hence, when n is even, $R(n)$ is maximum only if n is a power of 2.

If n is a power of 2,

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* This problem was suggested by R. Vaidyanathaswami.

$$\frac{\phi(n)}{n} = \frac{1}{2},$$

and $\phi(n)$ is again a power of 2. Hence,

$$2^{R(n)} = n.$$

Therefore,

$$R(n) = \frac{\log n}{\log 2}.$$

But, by (1), $\phi(n)$ cannot be odd when $n > 2$.

Hence, the maximum is attained when

$$n = p_1 p_2 \cdots p_s,$$

where $p_1, p_2, \cdots, p_s$ are primes of the form 2^l+1. Hence, if there are infinitely many primes of the form 2^l+1, then there are infinitely many n's for which

$$R(n) > \frac{\log n}{\log 2} + 1 - \epsilon$$

for every positive ϵ.

But even in this case,

$$R(n) \leqq \frac{\log n}{\log 2} + 1.$$

But there are infinitely many values of n for each of which $R(n) = \log n/\log 2$, namely, $n = 2^r$.

THEOREM II. *If n is such that $R(n) < R(n')$ for every $n' > n$, let us define n as H. Then*

$$H = 2 \cdot 3^r.$$

To prove this, we want two lemmas.

LEMMA 1. *If x is even,*

$$R(2x) = R(x) + 1.$$

The proof is obvious.

LEMMA 2. *Whether x is odd or even, a multiple of 3, or not,*

$$R(3x) = R(x) + 1.$$

provided that $x \geqq 2$.

Since $\phi(3) = 2$, this easily follows from Lemma 1.

Now we are in a position to prove Theorem II. Put

$$2m = m_1, \qquad m_1 p = n_1,$$

where p is an odd prime greater than 3. As long as $\phi_{r-1}(m_1)$ is divisible by p,

$$\frac{\phi_r(m_1)}{m_1} = \frac{\phi_r(n_1)}{n_1}.$$

Let $\phi_{r-1}(m_1)$ be divisible by p, and $\phi_r(m_1)$ not. Then, obviously,

$$\phi_r(n_1) = \phi_r(m_1) \times p$$

and

$$\phi_{r+1}(n_1) = \phi_{r+1}(m_1) \times (p-1) = 2\phi_{r+1}(m_1) \cdot \frac{p-1}{2} = 2 \cdot u \text{ (say)}.$$

Since $\phi_{r-1}(m_1)$ is divisible by the odd prime p which is >3, and $\phi_{r-1}(m_1)$ is even, $\phi_r(m_1)$ is even and is greater than 2. Hence, $\phi_{r+1}(m_1)$ is even. So, whether $(p-1)/2$ is odd or even, u is even. Therefore, by Lemma 1,

$$R(\phi_{r+1}(n_1)) = R(2u) = R(u) + 1.$$

But

$$\frac{\phi_{r+1}(n_1)}{\phi_r(n_1)} = \frac{\phi_{r+1}(m_1) \times (p-1)}{\phi_r(m_1) \times p} = \frac{2\phi_{r+1}(m_1)}{\phi_r(m_1)} \times \frac{p-1}{2p};$$

and

$$\frac{p-1}{2p} \geqq \frac{5-1}{2 \cdot 5} = \frac{4}{10} > \frac{1}{3}.$$

Hence, before the rate of decrease of $\phi_r(n_1)$ becomes 3 times that of $\phi_r(m_1)$, the value of $R(n_1)$ will be increased at least by one. This is the case at every stage where the rate of decrease is increased due to an odd prime in $\phi_r(n_1)$, which is not in $\phi_r(m_1)$. Hence, if

$$\frac{\phi_r(n_1)}{n_1} \leqq \frac{\phi_r(m_1)}{m_1} \times \frac{1}{3^l},$$

then

$$R(\phi_r(n_1)) \geqq R(\phi_r(m_1)) + l.$$

Hence we may write

$$\begin{aligned} R(n_1) &\geqq R(m_1) + \left[\frac{\log p}{\log 3}\right] + 1 \\ &= R(m_1) + k \text{ (say)} \\ &= R(3^k m_1), \end{aligned}$$

by Lemma 2. However, we have

$$3^k m_1 > n_1.$$

Therefore, n_1 cannot be an H.

Therefore, in order that n may be an H, n should be divisible by no odd prime greater than 3. Hence, if n is an H,

$$n = 2^s \cdot 3^t.$$

But

$$\phi_1(2^s \cdot 3^t) = 2^s \cdot 3^{t-1},$$

when $s \geqq 1$,

. . . . . .

and

$$\phi_t(2^s \cdot 3^t) = 2^s.$$

Therefore

$$R(2^s \cdot 3^t) = t + s.$$

Hence

$$R(2^s \cdot 3^t) = R(2 \cdot 3^{t+s-1}).$$

But

$$2 \cdot 3^{t+s-1} > 2^s \cdot 3^t.$$

Hence, unless $s < 2$, n is not an H. So, $n = 2 \cdot 3^t$ or 3^t. But

$$R(2 \cdot 3^t) = t + 1 = R(3^t).$$

Hence

$$n = 2 \cdot 3^t.$$

THEOREM III.

$$R(n) \geqq \left[\frac{\log n - \log 2}{\log 3}\right] + 1.$$

PROOF. By Theorem II, $R(n)$ is a minimum, when $n = 2 \cdot 3^t$, and then $R(n) = t + 1$.

But

$$t = \left[\frac{\log n - \log 2}{\log 3}\right].$$

Therefore

$$R(n) \geqq \left[\frac{\log n - \log 2}{\log 3}\right] + 1.$$

Hence

$$\left[\frac{\log n}{\log 2}\right] \geqq R(n) - 1 \geqq \left[\frac{\log \frac{n}{2}}{\log 3}\right].$$

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MULTIPLE POINTS OF ALGEBRAIC CURVES*

BY T. R. HOLLCROFT

1. *Introduction.* Limits to the number of multiple points of algebraic curves were first found by Cramer.† He found and tabulated the maximum numbers of multiple points of all possible orders for curves of orders up to and including eight. Plücker‡ obtained the general expression $(n-1)(n-2)/2$ for the maximum number of double points of an algebraic curve of order n.

Except for individual curves, the maximum number of multiple points of higher order than two for a curve of given order has not been found. A general expression for the maximum number of compound singularities or singularities of different orders is not practicable. When, however, the curve possesses only multiple points or sets of multiple points of the same order, serviceable limits for the maximum number of such singularities can be found.

The purpose of this paper is to determine the maximum number of distinct multiple points of given order and con-

* Presented to the Society, June 20, 1929.

† G. Cramer, *Introduction à l'analyse des lignes courbes algébriques*, Geneva, 1750, pp. 455–459.

‡ J. Plücker, *Theorie der Algebraischen Curven*, Bonn, 1839, p. 215.

secutive multiple points of given order that may belong to an algebraic curve of given order.

2. *Invariants Associated with Multiple Points.* With each singularity of a curve f there are associated certain functions of the coefficients of the equation of f, invariant under a projective transformation, whose vanishing is necessary and sufficient for f to possess this singularity. Such functions will be called invariants. Also, all curves herein considered are proper curves.

A necessary and sufficient condition for f to have a multiple point of order r at P is that all the partial derivatives of f up to and including the $(r-1)$st vanish at P. These partial derivatives constitute $r(r+1)/2$ linear functions of the coefficients of f. The postulation of an ordinary r-fold point on f is, therefore, $r(r+1)/2$. The two independent coordinates of P may be eliminated from the $r(r+1)/2$ relations among the coefficients in $r(r+1)/2-2$ independent ways giving rise to $r(r+1)/2-2$ invariants associated with each ordinary r-fold point.

Consecutive multiple points of f are multiple points that coincide at a point P in such a way that all the branches of f common to two or more of these multiple points have a common tangent at P. Enriques* has proved that the postulation of s consecutive multiple points of orders r_i on f is the same as the total postulation of the s multiple points considered as distinct on f, that is

$$\frac{1}{2}\sum_{i=1}^{s} r_i(r_i+1).$$

Assume that s points are consecutive on a curve. If one of the points P is fixed, one condition determines the direction of approach to P of each of the remaining $s-1$ points. The two parameters defining P and the $s-1$ parameters determining the $s-1$ directions through P total $s+1$ parameters.

* F. Enriques, *Lezioni sulla Teoria Geometrica delle Equazione e delle Funzioni Algebriche*, vol. 2, pp. 404–408.

Therefore the number of independent parameters involved in the location of s consecutive points on any plane curve is $s+1$.

The postulation of a singularity is the total number of conditions necessary and sufficient to determine both the nature of the singularity and its position. The $\sum r_i(r_i+1)/2$ relations among the coefficients of f then involve the $s+1$ parameters which determine the positions of the s consecutive points. From these $\sum r_i(r_i+1)/2$ relations the $s+1$ parameters may be eliminated in $\sum r_i(r_i+1)/2-s-1$ independent ways, each eliminant being an invariant associated with the singularity. There are, then,

$$\frac{1}{2}\sum_{i=1}^{s} r_i(r_i+1) - s - 1$$

invariants associated with s consecutive multiple points of orders r_i. These invariants determine the nature of this singularity of f and only that.

3. *The Maximum Number of Consecutive Multiple Points.* It has been proved by Lefschetz* that there can be no more that $n(n+3)/2-8$ invariants among the coefficients of the equation of an algebraic curve of order n and genus $p>1$. In the preceding section it has been found that $sr(r+1)/2-s-1$ invariants are associated with s consecutive r-fold points. A limit to the number of consecutive r-fold points that may occur on a curve of order n and genus $p>1$ is therefore determined by the inequality

$$\tfrac{1}{2}sr(r+1) - s - 1 \leqq \tfrac{1}{2}n(n+3) - 8$$

which, when solved for s, becomes

$$s \leqq \frac{n^2+3n-14}{(r-1)(r+2)}. \tag{1}$$

* S. Lefschetz, *On the existence of loci with given singularities*, Transactions of this Society, vol. 14 (1913), pp. 23–24.

For $r=2$, this is the limit to the number of consecutive nodes of a curve of order n found by Sharpe and Craig.*

Another limit to the number of consecutive r-fold points of a curve of order n is found from the fact that the total number of double points cannot exceed $(n-1)(n-2)/2$. Since an r-fold point contains $r(r-1)/2$ nodes, this gives the limit

$$s \leqq \frac{(n-1)(n-2)}{r(r-1)}. \tag{2}$$

For $r=2, 3, 4$, limit (1) holds for $n \geqq 6, 10, 13$, respectively, otherwise limit (2). Both limits give $s=10$ for $r=2$ and $n=6$. For $n \leqq 6$, therefore, all the nodes of an algebraic curve of any genus may be consecutive.

Limits (1) and (2) are equal for

$$r = \frac{(n-1)(n-2)}{3n-8}.$$

Limit (1) holds for all values of r less than or equal to the above value and limit (2) holds for all values of r greater than or equal to this value. However, there are certain exceptions to limit (2) now to be discussed.

Since limit (2) is determined by the maximum number of double points of the curve, this limit would permit a curve of order n to have more than one point of order $r \geqq (n+1)/2$. This is impossible, since a line joining two such points would intersect the curve in more than n points. The range of r for limit (2) is, therefore, restricted to $(n-1)(n-2)/(3n-8) \leqq r \leqq n/2$.

Within the above range for r, limit (2) holds except when the value of s determined by limit (2) is of the form $n'(n'+3)/2$, where n' is any positive integer. A curve of order n' through the s consecutive r-fold points of f cannot intersect f in more than nn' points. In case the value of s is not the exact number necessary to determine a curve of order

* Sharpe and Craig, *Plane curves with consecutive double points*, Annals of Mathematics, vol. 16 (1914–15), p. 19.

n', then additional simple points of f must be chosen to determine the curve of order n' through all the s consecutive r-fold points of f. In this case, the number of intersections of the two curves is always less than nn' when limit (2) is satisfied.

There remains, then, an exception to limit (2) for $r \leqq n/2$ only when limit (2) gives the value $s = n'(n'+3)/2$ and then only when $nn' < sr$. We shall now prove that in all cases when $n > 4$, $s = n'(n'+3)/2$ and $nn' < sr$, the inequality $sr - nn' < r$ holds.

Assume $s = n'(n'+3)/2 = (n-1)(n-2)/[r(r-1)]$ and eliminate n' from this equation and the inequality $sr - nn' < r$ to be proved. This gives

$$3nr^2 - r(n^2+6n-2) - (n-1)(n-2) < 0.$$

This inequality is evidently satisfied except for large values of r. The largest value of r occurs when $s = 5$. If we eliminate r from the equality $5r(r-1) = (n-1)(n-2)$ and the above inequality, there results an inequality in n which is satisfied for all values of $n > 4$.

Therefore, when the value of s defined by limit (2) is of the form $n'(n'+3)/2$ and $nn' < sr$, since also $sr - nn' < r$, the maximum number of consecutive r-fold points for that order n is one less than the value given by limit (2).

The least value of r for which limit (2) holds is $(n-1)(n-2)/(3n-8)$. Substitute this value of r in limit (2) and there results $s = 9$ for $n > 6$. Substituting the largest value $n/2$ of r in limit (2), we obtain $s = 3$. The entire range of values of s given by limit (2) for $n > 6$ is, therefore, $9 \geqq s \geqq 3$.

When $s = 9$, a cubic can be passed through the nine multiple points, that is, $n' = 3$. Since $(n-1)(n-2)/(3n-8) - n/3$ is less than $1/3$ for all values of $n > 6$, the largest value of r in proportion to n for which $s = 9$ occurs when n is a multiple of 3 and $r = n/3$. In this case, a cubic through the nine multiple points intersects f in $3n$ points, that is, all the intersections of f and the cubic occur at the multiple points. The only

restriction on the positions of the nine multiple points of order $n/3$ is that they can not all lie at the nine intersections of two cubics.

The only case in which $nn' < sr$, therefore, occurs when $n' = 2$ and $s = 5$. If we set $s = 5$ in limit (2), there results a value of r in terms of n such that five times this value of r is always greater than $2n$. Then, in accordance with the above proof, when $r > 2n/5$ and limit (2) gives $s = 5$, the maximum number of consecutive multiple points of this multiplicity is four. This is the only exception to limit (2) within the range of the values of r for which it holds.

Since limit (2) is defined by the maximum number of double points of a curve, this limit as well as the above discussion of it applies equally well to distinct multiple points and to consecutive multiple points.

When r has its maximum value for a given n, if n is even, $r = n/2$, $s = 3$ and if n is odd, $r = (n-1)/2$, $s = 4$ for $n > 5$. These multiple points may be either distinct or consecutive.

If k cusps, $0 \leqq k \leqq r-1$, replace the same number of nodes in s consecutive r-fold points, the limit for s obtained by considering the number of invariants associated with the singularity is

$$s \leqq \frac{n^2 + 3n - 14 - 2k}{(r-1)(r+2)}. \tag{3}$$

Limit (2) is, however, unchanged. For given values of r, k and n, the smaller of the two limits (2) and (3) now determines the maximum value of s. In case the inequality $k \geqq 3n - 8 - (n-1)(n-2)/r$ is satisfied, limit (3) holds, otherwise limit (2). The same restrictions on limit (2) obtained above for $k = 0$, hold for all values of k since in limit (2) and in determining the number of intersections of two curves, a cusp counts merely as a double point.

If a curve f of order n has j singularities each consisting of s consecutive nodes, the following limits are found:

(a) $$j \leqq (n^2 + 3n - 16)/(4s - 2),$$

(b) $$j \leqq (n-1)(n-2)/(2s).$$

For $s \leqq (n-1)(n-2)/[(n-4)(n-5)]$ limit (a) holds, otherwise limit (b).

For $s=2$, solve $(n-1)(n-2)/[(n-4)(n-5)] \geqq 2$ for n and we obtain $n \leqq 11$. For $n=12$, however, 54 of its possible 55 nodes may form 27 tacnodes and limit (a) still be satisfied. Then for $n \leqq 12$, a curve of any genus may have all its nodes consecutive in pairs to form tacnodes when the number of nodes is even or all but one when the number is odd. Similarly, for $n \leqq 9$ a curve of any genus may have as many oscnodes as the integral number of times three is contained in the number of its nodes.

If each of the singularities consists of $s-1$ nodes and one cusp, all consecutive, each singularity accounts for $2s$ invariants. Then the maximum number j of such singularities must satisfy limit (b) and also the limit

$$(a') \qquad j \leqq (n^2 + 3n - 16)/(4s).$$

Limit (a′) is less than limit (b) for $n \geqq 6$ and all values of s. For example, a quintic may have three ramphoid cusps or two tacnode-cusps. A sextic may have three tacnode-cusps, but not five ramphoid cusps nor two sets each containing four nodes and one cusp.

More generally, in order that a curve of order n have j singularities each consisting of s consecutive r-fold points, the following inequality must be satisfied:

$$j \leqq (n^2 + 3n - 16)/[s(r-1)(r+2) - 2],$$

and also the total number of nodes in the j singularities must not exceed $(n-1)(n-2)/2$.

In order to show that a curve of order n can possess a singularity consisting of s consecutive multiple points of orders $r_1, r_2, \cdots, r_s$, both of the following inequalities must be satisfied:

$$\sum r_i(r_i+1) - 2s \leqq n(n+3) - 14,$$
$$\sum r_i(r_i-1) \leqq (n-1)(n-2),$$

and also it must be shown that the sum of the $n'(n'+3)/2$

largest values of r_i does not exceed nn' for any value of n' such that $s \geqq n'(n'+3)/2$.

None of the above limits hold when more than two multiple points are collinearly consecutive. Since a line can not intersect a curve of order n in more than n points, the limit to the number of collinearly consecutive r-fold points is $s \leqq n/r$, or in case the multiple points are of different orders, the inequality $\sum r_i \leqq n$ must be satisfied.

4. *Limits for Distinct Multiple Points.* Since each r-fold point involves $r(r+1)/2-2$ invariants, the following limit defines the maximum number s of distinct r-fold points:

$$(1') \qquad s \leqq (n^2 + 3n - 16)/(r^2 + r - 4).$$

It was noted in the preceding section that limit (2) applies to distinct as well as to consecutive multiple points. Then limits (1′) and (2) apply to distinct multiple points. The discussion of limit (2) in the preceding section also applies equally well to distinct multiple points.

Limit (1′) holds for

$$r \leqq \{n^2 - 7 + [(n^2 - 7)^2 - 24(n-1)(n-2)(n-3)]^{1/2}\}/[6(n-3)],$$

otherwise limit (2). The radicand is negative for $n \leqq 16$, but for a given value of r the least value of n for which limit (1′) is less than limit (2) is 18.

For $r=2$, limit (1′) is always greater than limit (2). For $r=3$ or 4, limit (2) holds for $n \leqq 17$ and limit (1′) for $n \geqq 18$. As r increases, this discriminating value of n increases, for example, when $r=5$ or 6, limit (1′) holds for $n \geqq 20$, 23 respectively, otherwise limit (2).

We shall now prove that neither limit (1) for the maximum number of consecutive multiple points of order r nor limit (1′) for the maximum number of distinct multiple points of order r is subject to any restriction due to intersections of f with a curve through its multiple points.

For $r[(4n^2+12n-47)^{1/2}-1]/2$, limit (1′) $\geqq$ limit (1). For

$n \geqq 4$, this value of r is greater than the upper limiting value of r for which limit (1) holds. Then limit (1′) is larger than limit (1) for all values of n and r for which limit (1) holds. Then to prove that limit (1′) is never so large as to allow a curve of order n' through the s multiple points to have more than nn' intersections with the curve of order n possessing the multiple points will also establish the same property for limit (1).

Let $s=(n^2+3n-16)/(r^2+r-4)=n'(n'+3)/2$, that is, assume the largest possible value of s given by limit (1′) and assume further that this is the exact number of points necessary to determine a curve of order n' through the s distinct multiple points of order r. These are the conditions under which a curve of order n' through the s multiple points intersects the curve possessing the multiple points in the greatest number of points.

Together with the above equality, assume $nn' \geqq sr$. Eliminate n' from this inequality and the above equality and there results

$$r^2(n^2 + 3n - 16) - n(2n - 3r)(r^2 + r - 4) \leqq 0.$$

Substitute in this inequality the maximum value of r in terms of n for which formula (1′) holds and the resulting inequality in n is satisfied by all values of $n \geqq 5$.

Then the maximum number of multiple points of order r as defined by limit (1′) when distinct, or by limit (1) when consecutive, satisfies the criterion that no proper curve of order n' can be described through them intersecting the curve of order n possessing the multiple points in more than nn' points.

WELLS COLLEGE

CONTINUOUS CURVES IN WHICH EVERY ARC MAY BE EXTENDED*

BY W. L. AYRES†

1. *Introduction.* An arc α belonging to a point set K is said to be a *maximal arc* of K if there is no arc which belongs to K and contains the arc α as a proper subset. S. Mazurkiewicz‡ proved in 1921 that every arc of a bounded acyclic continuous curve§ M is a subset of a maximal arc of M. However there are many types of continuous curves in which every arc of a continuous curve is a subset of a maximal arc and which are not acyclic; and in a recent paper‖ we have given a necessary and sufficient condition in order that a continuous curve have this property. In this note we shall treat the opposite problem of finding under what conditions no arc is a subset of a maximal arc. We shall say that an arc α of a point set K is *extendible* in K if there exists an arc β of K of which α is a proper subset. If X and Y are the end points of α and Y is an interior point of β, then we say that α is *extendible in K in the direction XY*. And if there exists an arc β of K containing α and such that both end points of α are interior points of β, then we say α is *extendible in K in both directions*. In this note we shall give two necessary and suffi-

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‡ *Un théorème sur les lignes de Jordan*, Fundamenta Mathematicae, vol. 2 (1921), pp. 119–130. See Lemma 13, p. 129.

§ A *continuous curve* is a closed connected, and connected im kleinen point set. A continuous curve is *acyclic* if it contains no simple closed curve. It must be noticed that under these definitions a single point is a continuous curve and is acyclic. However, to avoid trivial cases, we will assume that the continuous curve M, which is mentioned repeatedly, contains more than one point.

‖ W. L. Ayres, *Conditions under which every arc of a continuous curve is a subset of a maximal arc of the curve*, Mathematische Annalen, vol. 101 (1929), pp. 194–209.

cient conditions that every arc of a continuous curve M be extendible in M in both directions. The first of these conditions is that M shall not contain a certain type of point. From this it may be seen that in order that every arc of M be merely extendible in M it is necessary and sufficient that M contain at most one point of this type.

All of the point sets mentioned in this paper are assumed to lie in a two-dimensional euclidean space E_2.

2. *Preliminary Lemmas.* P. Urysohn* has shown that every point of a continuous curve may be ϵ-separated,† for any positive number ϵ, by a set consisting of a finite number of continua. By a result due to G. T. Whyburn and the author,‡ each of these continua may be replaced by a continuous curve so that we still have an ϵ-separation. Hence in characterizing types of points of a continuous curve through their ϵ-separating sets we may consider only the sets of continuous curves which form an ϵ-separating set. In the plane we may show that we need consider only arcs§ and simple closed curves.

LEMMA A. *If P is a point of a continuous curve M and ϵ is a positive number, there exists a simple closed curve J of*

* *Ueber im kleinen zusammenhängende Kontinua*, Mathematische Annalen, vol. 98 (1927), p. 297.

† A set B is said to be an *ϵ-separation* of the point P in the set M if $M-B=A+C$, where A and C are mutually separated sets and A contains the point P, and further the diameter of the set $A+B$ is less than ϵ. See P. Urysohn, *Mémoire sur les multiplicités Cantoriennes*, Fundamenta Mathematicae, vol. 7 (1925), p. 65.

‡ *On continuous curves in n dimensions*, this Bulletin, vol. 34 (1928), pp. 349–360, Theorem 1. For two quite similar theorems published at approximately the same time, see P. Alexandroff and L. Tumarkin, *Beweis des Satzes, dass jede abgeschlossene Menge positiver Dimension in einem lokal zusammenhängenden Kontinuum von derselben Dimension topologisch enthalten ist*, Fundamenta Mathematicae, vol. 11 (1928), pp. 141–144; and W. Stepanoff and L. Tumarkin, *Ueber eine Erweiterung abgeschlossener Mengen zu Jordanschen Kontinuen derselben Dimension*, Fundamenta Mathematicae, vol. 12 (1928), pp. 43–46. However for the present application we require the property (2) of the theorem by Whyburn and myself. This property is stated in neither of the latter two theorems.

§ We consider a point as a special case of an arc.

diameter less than ϵ, enclosing P and such that $J \cdot M$ consists of a finite number of components, that is, every point of M may be ϵ-separated by a simple closed curve or a finite set of mutually exclusive arcs. If there exists a positive number r and a positive integer n such that, for every circle C with center P and radius less than or equal to r, the point P belongs to the boundary of exactly n $(n>0)$ complementary domains of $M>C$, then P may be ϵ-separated in M by a set of n arcs, for any positive number ϵ.

If P belongs to the boundary of no complementary domain of M, then M contains a simple closed curve which encloses P and is of diameter less than ϵ*. If P belongs to the boundary of a complementary domain D of M, let C be a circle with center P and radius less than $\epsilon/2$ and such that C contains a point of D. Now using a slight modification of the proof of Theorem 1 of our paper, *On the separation of points of a continuous curve by arcs and simple closed curves*,† we may complete our proof.

LEMMA B. *If the point P of the continuous curve M may be ϵ-separated, for any positive number ϵ, by a set S of n mutually exclusive continua such that P is not contained in a bounded complementary domain of any one, but not by any set S of less than n, then P may be ϵ-separated by a set of n mutually exclusive arcs.*

For each positive integer k, let C_k be a circle with center P and radius $1/k$ and let π_k be the number of complementary domains of $M+C_k$ that have P and a point of C_k on their boundaries. Evidently $\pi_{k+1} \geqq \pi_k$ for every k. If the number π_k increases indefinitely with k, then it is easy to see that the number of components of any ϵ-separating set must increase indefinitely as ϵ approaches zero. But by hypothesis, n components suffice. Hence there is an integer s such that for $t \geqq s$, $\pi_t = \pi_s$. First let us suppose that $\pi_s > 0$.

* G. T. Whyburn, *Concerning continua in the plane*, Transactions of this Society, vol. 29 (1927), p. 381, Theorem 7.

† Proceedings of the National Academy of Sciences, vol. 14 (1928), pp. 201–206.

It follows from Lemma A that P may be ϵ-separated by a set of π_s mutually exclusive arcs for any $\epsilon>0$. By our hypothesis, $\pi_s \geqq n$. And it is easy to see that for $\epsilon<1/s$, no ϵ-separating set has less than π_s components. But P may be ϵ-separated by a set of n continua for each ϵ. Therefore $\pi_s \leqq n$. Hence $\pi_s = n$ and our lemma is proved for the case $\pi_s>0$.

Now consider $\pi_s=0$. There are two cases according as P is or is not a limit point of E_2-M. If P is not a limit point of E_2-M, there exists a positive number r such that every point of the interior of a circle with center P and radius r belongs to M. Then for $\epsilon<r/2$ there is no ϵ-separating set S such that P is not contained in a bounded complementary domain of some one component of S. Hence this case is impossible. In the remaining case we shall show that n must be equal to one and complete our proof by establishing the following lemma.

LEMMA C. *If B denotes the point set consisting of the sum of the boundaries of the complementary domains of a continuous curve M, and P is a point of $\overline{B}-B$, then P may be ϵ-separated in M by a single arc, for any positive number ϵ.*

There exists a simple closed curve J_1 which belongs to M, is of diameter less than ϵ and contains P in its interior. As P belongs to $\overline{B}-B$ there exists a complementary domain D of M in the interior of J_1 and a positive number ϵ' such that no point of J_1+D is within a distance ϵ' of P. There exists a simple closed curve J_2 which belongs to M, is of diameter less than ϵ', and contains P in its interior.* Then J_2 lies in the interior of J_1 and D lies between J_1 and J_2. Now let J_3 be any simple closed curve which lies between J_1 and J_2 and contains a point of D. Let $G_1, G_2, G_3, \cdots$ be the set of all components of $J_3 \cdot (E_2-M)$. This set contains at least one component and is countable. Each component G_i lies in a complementary domain D_i of M. Let B_i be the boundary of D_i and let

$$G = (J_3 - \sum G_i) + \sum B_i.$$

* G. T. Whyburn, *Concerning continua in the plane*, loc. cit.

Now (1) G lies entirely between J_1 and J_2 except for points on J_1 and J_2, (2) M contains G, (3) G is a continuous curve, (4) P lies in a bounded complementary domain of G.* The outer boundary of this domain is a simple closed curve J_4† which belongs to M, contains P in its interior, and has two points X and Y on the boundary of some complementary domain D_i. Let XUY be an arc from X to Y which lies in D_i except for X and Y. The point P lies in the interior of J_4 and thus it lies in the interior of one of the two simple closed curves formed by the arc XUY plus one of the arcs of J_4 from X to Y. Call this arc XVY. Then XVY is the desired ϵ-separating arc, for P lies in the interior of the simple closed curve $XUY+XVY$ and this simple closed curve lies in the set consisting of J_1 plus its interior.

3. THEOREM 1. *In order that every arc of a continuous curve M be extendible in M in both directions it is necessary and sufficient that M contain no point which can be ϵ-separated in M by an acyclic continuous curve for every positive number ϵ.*

The condition is sufficient. Every "point" arc may be extended because every point of a continuous curve, except end points, is interior to an arc of M. But an end point may be ϵ-separated by a point for every ϵ, and there exists no such point by our condition. Now let α be any true arc of M and let P and Q be the end points of α. We shall show that α is extendible in M in the direction QP. Suppose this is not the case. Let us consider the components of $M-\alpha$. No one of these has P as a limit point for then α would be extendible in M in the direction QP, contrary to our supposition. Let ϵ be any positive number. There exists a point X of α so that the diameter of the subarc XP of α is less than ϵ, and there exists a point Y ($X \neq Y$) of the subarc XP so that

* For detailed proof of these and the following statements, see page 204 of my paper, *On the separation of points of a continuous curve by arcs and simple closed curves*, loc. cit.

† R. L. Moore, *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), pp. 254–260, Theorem 4.

the diameter of the subarc YP of α is less than $\epsilon/2$. Since only a finite number of the components of $M-\alpha$ are of diameter greater than any given positive number,* there exists a point $Z(Z \neq Y)$ on the subarc YP of α so that (1) no component of $M-\alpha$ of diameter greater than $\epsilon/2$ has a limit point on ZP and (2) no component of $M-\alpha$ has a limit point on both ZP and XQ. Then the subarc XZ is an ϵ-separating set of P in M. But this is contrary to the hypothesis. Hence α is extendible in M in the direction QP, and similarly in the direction PQ.

The condition is necessary. Suppose M contains a point P which can be ϵ-separated by an acyclic continuous curve for every ϵ. We shall show that this leads to a contradiction by constructing in M an arc which cannot be extended in M in one direction. By Lemma B, P may be ϵ-separated in M by an arc. There exists a sequence $J_1, J_2, J_3, \cdots$ of simple closed curves such that (1) the interior of J_i contains J_{i+1} and P, (2) the diameter of J_i approaches zero as i increases indefinitely, (3) $J_i \cdot M = A_iB_i$ consists of a single arc. When A_iB_i is not a point, then A_i and B_i are the end points of A_iB_i. In the order from A_i to B_i let C_i and D_i denote the first and last limit points of the component of $M-A_iB_i$ that contains P. Let E_i and F_i $(i>1)$ denote the first and last limit points on A_iB_i of that component of $M-A_iB_i$ that contains $A_{i-1}B_{i-1}$. Each of the subarcs C_iD_i and E_iF_i of A_iB_i is a cutting of M. The component of $M-C_iD_i$ containing P lies in the interior of J_i, and the component of $M-E_iF_i$ containing P lies in the interior of J_{i-1}. There exists an arc α with end points D_1 and D_2 and lying except for D_1 entirely in the component of $M-A_1B_1$ that contains P.† In the order from D_1 to D_2 there is a first point G_1 of A_2B_2 on α. Consider the two arcs C_3D_3 and E_3F_3. Either one end point of E_3F_3 is not an interior point of C_3D_3 or both end

* W. L. Ayres, *Note on a theorem concerning continuous curves*, Annals of Mathematics, vol. 28 (1927), pp. 501–2.

† R. L. Wilder, *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), p. 342.

points of E_3F_3 are interior points of C_3D_3. In the first case let H_1 be this end point. And in this case there is an end point H_2 of C_3D_3 so that the subarc H_1H_2 of A_3B_3 contains the entire arc C_3D_3. In the latter case let H_1 be either end point of E_3F_3 and let H_2 be that end point of C_3D_3 so that the subarc H_1H_2 of A_3B_3 contains the entire arc E_3F_3. In either case, then, we have a subarc H_1H_2 of A_3B_3 so that H_1H_2 contains either C_3D_3 or E_3F_3. There exists* an arc G_2H_1 of M which lies, except for one end point G_2 on J_2 and one end point H_1 on J_3, entirely between J_2 and J_3. In general, the arc $A_{2n+1}B_{2n+1}$ $(n=1, 2, 3, \cdots)$ contains an arc $H_{2n-1}H_{2n}$ which contains one of the two arcs $C_{2n+1}D_{2n+1}$ or $E_{2n+1}F_{2n+1}$, and is such that H_{2n-1} is either E_{2n+1} or F_{2n+1} and H_{2n} is either C_{2n+1} or D_{2n+1}. There exist two arcs $G_{2n}H_{2n-1}$ and $H_{2n}G_{2n+1}$ of M such that G_{2n} is a point of J_{2n}, G_{2n+1} is a point of J_{2n+2}, and these two arcs lie except for their end points entirely between J_{2n} and J_{2n+1} and J_{2n+1} and J_{2n+2} respectively. Let

$$\beta = P + C_1D_1 + \text{subarc } D_1G_1 \text{ of } \alpha + \sum_{n=1}^{\infty}(\text{subarc } G_{2n-1}G_{2n} \text{ of } A_{2n}B_{2n}) + \sum_{i=1}^{\infty}H_{i+1}G_{i+1} + \sum_{n=1}^{\infty}(\text{subarc } H_{2n-1}H_{2n} \text{ of } A_{2n+1}B_{2n+1}).$$

Evidently β is an arc of M from C_1 to P. And further we may show that this arc may not be extended in M in the direction C_1P for each of the arcs $H_{2n-1}H_{2n}$ contains a cutting of M and these cuttings of M are such that the diameter of the component of $M-H_{2n-1}H_{2n}$ containing P approaches zero as n increases. Hence any arc of M with P as an end point contains a point of infinitely many of the arcs $H_{2n-1}H_{2n}$. Therefore β cannot be extended in M in the direction C_1P.

4. THEOREM 2. *In order that every arc of the continuous curve M be extendible in M in both directions it is necessary and sufficient that* (1) *the sum B of the boundaries of all the complementary domains of M be a closed point set,* (2) *every point of B be either a cut point of M or belong to the boundaries of at least two complementary domains of M.*

* See R. L. Wilder's paper cited in the footnote on p. 855.

The conditions are necessary. Suppose $\overline{B}-B$ contains a point P. By Lemma C, P may be ϵ-separated in M by an arc for every positive ϵ. But by Theorem 1, M contains no such point P. Therefore $\overline{B}-B$ is vacuous and the first condition is necessary. Now suppose P is any point of B. If C is any circle with center P, then P is accessible from at least one complementary domain of $M+C$ since B contains P. If for every circle C, P is accessible from just one complementary domain of $M+C$, then P may be ϵ-separated, for any $\epsilon>0$, by an arc, from Lemma A. But this is impossible by Theorem 1. Hence there is some circle C so that P is accessible from at least two complementary domains D_1 and D_2 of $M+C$. If D_1 and D_2 belong to different complementary domains of M, we have the latter part of condition (2). And if D_1 and D_2 belong to the same complementary domain of M, it may be shown that P is a cut point of M, which is the first part of condition (2).

The conditions are sufficient. Let P be a point of M. If P is not a point of B, it is not a point of $\overline{B}$ by condition (1) and there exists a circle with center P and radius r such that every point of the interior of the circle belongs to M. Then P cannot be ϵ-separated by an acyclic continuous curve for $\epsilon<r$. If P belongs to the boundary of just one complementary domain of M, P is a cut point of M and for ϵ sufficiently small, no cut point may be ϵ-separated in M by a set consisting of just one component. And if P belongs to the boundaries of two complementary domains of M, we may show that any ϵ-separating set, for ϵ sufficiently small, must contain more than one component. Hence M contains no point which can be ϵ-separated in M, for every positive ϵ, by an acyclic continuous curve. By Theorem 1, every arc of M is extendible in M in both directions.

5. THEOREM 3. *If every arc of the continuous curve M is extendible in M in both directions, every point of M lies on the boundary of some complementary domain of M* $(M\equiv B)$.

By Theorem 2, the set B, the sum of the boundaries of the

complementary domains of M, is a closed point set. Now suppose there is a point P of $M-B$. Every complementary domain of M is a complementary domain of the point set B and in addition there is at least one complementary domain of B, all of whose points belong to M. Let D denote the complementary domain of B containing P. We shall assume that D is bounded for we may always produce this by an inversion. Let B_1 denote the outer boundary of D.* Then B_1 is the common boundary of two domains, one the exterior of B_1 and the other a bounded domain containing D.† Then no point of B_1 disconnects B_1.‡ Then by a theorem due to R. L. Moore,§ only a countable number of the points of B_1 are cut points of M. Hence, by Theorem 2, uncountably many points of B_1 are accessible from two complementary domains of M. Each of these points is also a limit point of D. There are only a countable number of complementary domains of M and thus only a countable number of pairs of complementary domains of M. And by a theorem due to G. T. Whyburn,¶ at most two points can be accessible from each of the two domains of a pair and at the same time be limit points of D. Thus in all only a countable number of points of B_1 can be accessible from two complementary domains of M. Therefore our assumption that $M-B$ is non-vacuous is false.

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* R. L. Moore, loc. cit., p. 256.

† Ibid., Theorem 3, p. 258.

‡ R. L. Moore, *Concerning the common boundary of two domains*, Fundamenta Mathematicae, vol. 6 (1924), pp. 203–213, Theorem 2.

§ *Concerning the cut-points of continuous curves and of other closed and connected point-sets*, Proceedings of the National Academy of Sciences, vol. 9 (1923), pp. 101–106, Theorem B*.

¶ *Concerning plane closed point sets which are accessible from certain subsets of their complements*, Proceedings of the National Academy of Sciences, vol. 14 (1928), pp. 657–666, Theorem 1 and lines 1–3 of page 659.

CLASSES OF DIOPHANTINE EQUATIONS WHOSE POSITIVE INTEGRAL SOLUTIONS ARE BOUNDED*

BY D. R. CURTISS

1. *Introduction.* If upper bounds have been found for the positive integral solutions of a diophantine equation, the problem of obtaining all such solutions is reduced to making a finite number of trials. It may therefore be of interest to note certain cases where upper bounds are given by simple algegraic processes. Hereafter the term *solution* will always mean a solution in positive integers.

Our starting point is the observation that if $P(t)$ is a polynomial in t, then all positive values of x that satisfy the inequality $P(1/x) \geqq 0$ are bounded if (and only if) the term of lowest degree in $P(t)$ has a negative coefficient.

2. *A Type whose Solutions are always Bounded.* Every algebraic diophantine equation in n variables $x_1, x_2, \cdots, x_n$ can be thrown into a form where the right side is zero and the left side is a polynomial in the reciprocals of the x's. When this has been done, the first type here to be considered is the following:

$$F\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n}\right) - k = 0, \tag{1}$$

where F is a polynomial all of whose coefficients are positive, while k is a positive constant, and $F(0, 0, \cdots, 0) = 0$.

The positive integral solutions of every equation of type (1) *are bounded.*

To prove this statement, and to show how to obtain bounds for the solutions, let us first consider a solution such that $x_1 \leqq x_2 \leqq \cdots \leqq x_n$. We shall then have

* Presented to the Society, December 28, 1927.

$$F\left(\frac{1}{x_1}, \frac{1}{x_1}, \cdots, \frac{1}{x_1}\right) - k \geqq 0, \tag{2}$$

since each term of F is not decreased when another x is replaced by x_1. The term of lowest degree in (2), $-k$, is negative, hence, from the observation made in the second paragraph of this paper, x_1 must be bounded. To obtain an explicit upper bound we note that, since $1/x_1^n \leqq 1/x_1$ when n and x_1 are positive integers, and since all the terms of F are positive, we have

$$F\left(\frac{1}{x_1}, \frac{1}{x_1}, \cdots, \frac{1}{x_1}\right) \leqq \frac{1}{x_1} F(1,1, \cdots, 1).$$

Hence when this result is applied to (2) we obtain

$$\frac{1}{x_1} F(1,1, \cdots, 1) - k \geqq 0,$$

or

$$x_1 \leqq \frac{1}{k} F(1,1, \cdots, 1). \tag{3}$$

Usually a lower bound than this can be derived by finding a closer approximation for the (unique) positive root of the equation obtained from (2) by retaining only the sign of equality.

Let us now find a bound for $x_r (r \leqq n)$ when upper bounds X_j have been assigned for each x_j from $j=1$ to $j=r-1$ in a solution where the x's form an ascending sequence from x_1 to x_n. We write (1) in the form

$$\begin{aligned} &F\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n}\right) \\ &- F\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_{r-1}}, 0, 0, \cdots, 0\right) \\ &= k - F\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_{r-1}}, 0, 0, \cdots, 0\right). \end{aligned} \tag{4}$$

The first two lines of (4) reduce to a polynomial

$$F_r\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_{r-1}}, \frac{1}{x_r}, \frac{1}{x_{r+1}}, \cdots, \frac{1}{x_n}\right)$$

each of whose terms is positive and involves at least one of the variables $x_r, \cdots, x_n$. Its value will not be decreased if each of the variables $x_{r+1}, \cdots, x_n$ is replaced by x_r. The last line of (4) is positive for all x's from x_1 to x_{r-1} that belong to solution systems, since the F function that appears here consists only of certain terms of the complete F in equation (1) and must lack some of the terms* of the complete F when $r-1<n$. Hence the inequality derived from (4),

$$F_r\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_{r-1}}, \frac{1}{x_r}, \frac{1}{x_r}, \cdots, \frac{1}{x_r}\right) - \left[k - F\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_{r-1}}, 0, 0, \cdots, 0\right)\right] \geqq 0,$$

is of the type $P(1/x_r) \geqq 0$, with term of lowest degree in $1/x_r$ negative. It follows that x_r is bounded for each set of values of the preceding x's. If these preceding x's are bounded, there must be an upper bound for all the values of x_r that belong to solution systems. Since we have shown that x_1 is bounded, it follows that all the x's are bounded, in solutions where $x_1 \leqq x_2 \leqq \cdots \leqq x_n$. We conclude at once that they are bounded for every order of relative magnitudes.

If the x's are arranged in order of magnitude from x_1 to x_n, an explicit upper bound for x_r, of which that given by (3) for x_1 is a special case, is indicated by the inequality

$$x_r \leqq \frac{1}{m_r} F_r(1, 1, \cdots, 1) = X_r, \tag{5}$$

where m_r is the least positive value of

$$k - F(1/x_1, 1/x_2, \cdots, 1/x_{r-1}, 0, 0, \cdots, 0)$$

for $x_j \leqq X_j$, $(j=1, 2, \cdots, r-1)$. Since m_r may be difficult to

* We assume that each of the variables $x_1, \cdots, x_n$ is explicitly present in the F function of equation (1).

evaluate, we note that (5) can be replaced by the weaker inequality

$$x_r \leqq D_r F_r(1,1,\cdots,1) = X_r', \tag{6}$$

where D_r is the product of all the denominators of terms of

$$k - F(1/X_1', 1/X_2', \cdots, 1/X_{r-1}', 0, 0, \cdots, 0).$$

As an illustration, consider the equation

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = 1.$$

Here $F(1, 1, \cdots, 1) = n$, and $F_r(1, 1, \cdots, 1) = n-r+1$. The value of $1/m_r$ is u_r, where $u_1 = 1$, $u_{k+1} = u_k(u_k+1)$.* Thus from (5) we obtain the upper bounds

$$X_r = (n - r + 1)u_r, \quad (r = 1,2, \cdots, n).$$

From (6) we have another set of bounds X_r', such that

$$X_r' = (n - r + 1)X_1'X_2' \cdots X_{r-1}'.$$

Hence

$$X_1' = n, \; X_2' = (n-1)n,$$
$$X_r' = (n-r+1)(n-r+2)(n-r+3)^2(n-r+4)^4 \cdots n^{2^{r-2}}, r>1.$$

3. *More General Types.* Equations of more general type than (1) can be written in the form

$$F\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n}\right) = G\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n}\right), \tag{7}$$

where F and G are polynomials all of whose coefficients are positive; we suppose all possible cancellations to have been performed. One or more of the variables may not be present in F, and the same may be true of G, but no variable is to be absent from $F-G$. We now investigate bounds for solutions where $x_1 \leqq x_2 \leqq \cdots \leqq x_n$.

If there is a constant term on either side of (7), x_1 *is bounded.*

* See *On Kellogg's diophantine problem*, American Mathematical Monthly, vol. 29 (1922), p. 380.

For, if, for example, we have $G(0, 0, \cdots, 0) = k \neq 0$, then $F(1/x_1, 1/x_1, \cdots, 1/x_1) \geqq k$, and formula (3) gives a bound for x_1. If (7) has no constant term, we consider the two inequalities derived from (7),

$$F\left(\frac{1}{x_1}, \frac{1}{x_1}, \cdots, \frac{1}{x_1}\right) - G\left(\frac{1}{x_1}, 0, \cdots, 0\right) \geqq 0,$$

$$G\left(\frac{1}{x_1}, \frac{1}{x_1}, \cdots, \frac{1}{x_1}\right) - F\left(\frac{1}{x_1}, 0, \cdots, 0\right) \geqq 0.$$

If the term of lowest degree in $1/x_1$ on the left of either of these inequalities is negative, we conclude that x_1 is bounded.

Again, using the notation of the earlier part of this paper, we deduce from (7) the inequality

$$(8) \qquad F_r\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_{r-1}}, \frac{1}{x_r}, \frac{1}{x_r}, \cdots, \frac{1}{x_r}\right) - \left[G\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_{r-1}}, 0, 0, \cdots, 0\right) - F\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_{r-1}}, 0, 0, \cdots, 0\right)\right] \geqq 0,$$

and another in which F_r is replaced by G_r, and F and G are interchanged. Unless the expression in brackets is zero, one of these inequalities is of the type $P(1/x_r) \geqq 0$, with negative constant term. Hence *unless the pair of equations*

$$(9) \qquad F\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n}\right) = G\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n}\right),$$

$$F\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_{r-1}}, 0, 0, \cdots, 0\right) = G\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_{r-1}}, 0, 0, \cdots, 0\right)$$

has a solution in positive integers $x_1, x_2, \cdots, x_n$, arranged in that order of magnitude, every solution of (7) *so ordered has x_r bounded if all the preceding x's are bounded.* From this we at

once obtain a sufficient condition that all the x's be bounded.

If the above condition fails we may replace the expression in brackets in (8) by

$$G\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_r}, 0, 0, \cdots, 0\right) - F\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_r}, 0, 0, \cdots, 0\right),$$

and F_r by F_{r+1}. If in the new inequality thus obtained the term of lowest degree in $1/x_r$ is negative, or if it is negative in the companion inequality, we infer again that x_r is bounded if this is true of the preceding x's.

An example to show that the above conditions may not be fulfilled is given by the equation

$$1 + \frac{1}{x_2} = \frac{1}{x_1} + \frac{1}{x_3^2}.$$

Here for $r=2$ the second equation of system (9) becomes

$$1 = \frac{1}{x_1},$$

and the pair of equations (9) is satisfied by $x_1=1$, $x_2=x_3^2$, where x_2 and x_3 are not bounded. On the other hand, many equations whose solutions are bounded escape these tests; an example is

$$\frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{x_1} - \frac{3}{x_2} = 0,$$

whose only solution in positive integers is $x_1=x_2=1$.

4. *Algebraic Equations with Positive Integral Roots.* A corollary of the first theorem of this paper concerns itself with algebraic equations

$$x^n - a_1x^{n-1} + a_2x^{n-2} - \cdots + (-1)^n a_n = 0,$$

all of whose roots are positive integers. There is, of course, an infinite number of such equations for any given n. But

there is only a finite number whose coefficients satisfy a relation

$$F\left(\frac{a_1}{a_n}, \frac{a_2}{a_n}, \cdots, \frac{a_{n-1}}{a_n}\right) = k,$$

where F is a polynomial with positive coefficients and $k>0$; for F is a polynomial in the reciprocals of the roots, and, when thus expressed, F has no constant term, so that the first theorem of this paper applies. We could obtain upper bounds for the roots, and therefore for the a's, by the methods of this paper. For example, if $a_{n-1}=a_n$, and if $x_1, x_2, \cdots, x_n$ are the roots, the x's must be solutions of the equation

$$\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}=1,$$

which has been discussed in § 2.

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ERRORS IN KRAITCHIK'S TABLE OF LINEAR FORMS

BY D. H. LEHMER

Tables of the linear forms that belong to a given quadratic residue D, or in other words, the linear divisors of the quadratic form t^2-Du^2 were first published by Legendre.* A list of errors in these fundamental tables has been given by D. N. Lehmer.† Kraitchik‡ has recalculated and extended these tables to the limit $D=\pm 250$. It is of great importance in using the table that every entry be correct. Therefore in constructing his factor stencils, D. N. Lehmer found it advisable to make a new table by means of a more or less graphical method.§ This table which has not been pub-

* *Théorie des Nombres*, 1st. ed., Tables III–VII, 1798.

† This Bulletin, vol. 8 (1902), p. 401. See also the correction in this Bulletin, vol. 31 (1925), p. 228.

‡ *Théorie des Nombres*, vol. 1, p. 164–186, Paris, 1922. *Recherches sur la Théorie des Nombres*, vol. 1, p. 205–215, Paris, 1924.

§ This Bulletin, vol. 31 (1925), pp. 497–498.

lished extends to $D = \pm 300$. The following list of errors results from collating these two tables. Kraitchik's forms for $D = \pm 182$ contain so many errors that a complete list of correct forms is given below. In as much as there were no errors detected in Lehmer's table it is reasonable to expect that the following list is complete.

ERRORS IN KRAITCHIK'S TABLE OF LINEAR DIVISORS

Théorie des Nombres, vol. 1.			*Théorie des Nombres*, vol. 1.		
D	For	Read	D	For	Read
+38	59	53	+157	107	109
−38	116	117	−157	471	529
−42	55	53	+165	112	113
−42	159	157	−166	473	477
+69	55	53	−173	655	309
−86	89	87	+174	203	61
−102	147	145	−181	359	357
−103	67	79	−181	491	461
−103	177	179	−181	719	721
−105	57	67	−185	661	253
−106	73	71	+190	119	197
−107	191	193	+191	173	175
−109	333	103	+191	271	275
−110	39	49	+193	155	129
−110	207	217	−193	541	155
−113	397	171	−193	617	231
+122	195	199	+194	41	47
−138	163	169	−194	453	455
−141	413	415	−197	191	199
+146	77	119	+199	309	257
−146	77	303	−199	309	257
−149	367	365	−199	insert	371
+151	183	189			

Recherches, vol. 1.			*Recherches*, vol. 1.		
D	For	Read	D	For	Read
+211	287	289	+230	23	33
−217	319	317	−233	915	925
−218	533	535	−241	607	357
+222	99	95	−241	697	693
−222	483	485	−241	731	733
−226	375	373	−246	387	389
−226	385	395	−247	105	449
−226	387	397	−249	197	695
+227	241	261	−249	301	799
−229	197	199			

CORRECT TABLES FOR $D = \pm 182$

$D = +182$

$728n \pm$						
	1	9	15	19	25	33
	37	41	43	51	55	59
	61	69	71	73	81	83
	85	87	89	93	97	101
	103	107	109	113	115	121
	135	141	145	149	151	155
	157	159	171	173	179	181
	187	197	199	201	211	225
	227	233	235	237	239	241
	253	265	269	285	289	297
	307	311	317	319	333	335
	337	341	347	353	359	361

$D = -182$

$728n +$						
	1	3	9	11	23	25
	27	31	33	37	41	47
	61	67	69	73	75	79
	81	85	89	93	95	97
	99	101	109	111	113	121
	123	127	131	139	141	145
	149	157	163	167	173	181
	183	191	197	201	207	215
	219	223	225	233	237	241
	243	251	253	255	263	265
	267	269	271	275	279	283
	285	289	291	295	297	303
	317	323	327	331	333	337
	339	341	353	355	361	363
	369	379	381	383	393	407
	409	415	417	419	421	423
	435	447	451	467	471	479
	489	493	499	501	515	517
	519	523	529	535	541	543
	549	551	557	563	569	573
	575	577	591	593	599	603
	613	621	625	641	645	657
	669	671	673	675	677	683
	685	699	709	711	713	723

BROWN UNIVERSITY

MENGER ON THEORY OF DIMENSIONS

Dimensionstheorie. By Karl Menger. Leipzig and Berlin, B. G. Teubner, 1928. iv+319 pp.

There is an important phase in the development of modern point set theoretical geometry which has been closely associated with the concept of dimensionality,—we refer to the attempt to create precise mathematical meaning for the simple geometric spaces of our intuition in terms of primitive non-arithmetical concepts. That the idea of dimensionality should have come into play and itself have been studied and made precise is indeed natural, since the curves, surfaces, and solids of our experience furnish the very basis for our intuitive ideas of dimensionality. The simple arithmetic definition of dimensionality, however, as the number of parameters required to define a space, while useful in ordinary geometry, was of course entirely inadequate when the more abstract spaces came into consideration, and it became highly desirable that dimensionality be relieved of its arithmetical vesture and be based on the inner structure of space itself.

Consider the spaces of our experience. What non-arithmetic relations among them are intuitively certain? The following immediately suggest themselves: a solid can be separated into several parts by one or more surfaces, a surface by curves and a curve by points. It was Poincaré who in 1912 suggested that precisely this type of phenomenon might lead to a satisfactory non-arithmetic definition of dimensionality,—a definition by recurrence. A space may be called n-dimensional, he suggested, if it can be separated into several parts by means of continua of $n-1$ dimensions.

Although the Poincaré definition was far from satisfactory either in precision or in content, it must be regarded as of the highest historical importance, since it indicated the possibilities of definition by recurrence, and was moreover essentially topological. These virtues were recognized by Brouwer, who in 1913 slightly modified the content of the definition and stated it in terms of the topologically precise notions of separation and connectedness; as a basis for recurrence, a zero-dimensional space was defined to be one which contained no continuum as subset. Brouwer showed that this "natural" definition of dimensionality satisfied the formal requirement of yielding the number n when applied to a cartesian S_n.

The dimensionality definition has since undergone further modification. If dimensionality was to be studied per se, it was of course desirable to arrive at a definition which would give rise to a theory of the highest generality and simplicity. To this end the basis for recurrence was altered, and what was of more significance, the concept of dimensionality as a local property was introduced. In its modern form the definition is as follows: a space is at most n-dimensional if each point is contained in an arbitrarily small neighborhood with a boundary of dimensionality at most $n-1$; (-1)-dimensional spaces are null spaces. A space which fails to be at most $(n-1)$-dimensional is at least n-dimensional. A space is n-dimensional if it

is at most and at least n-dimensional. Finally, an obvious modification yields the definition of dimensionality at a point.

An equivalent to this definition was introduced some eight years ago by the youthful Russian mathematician Paul Urysohn, who developed his theory in one of the most brilliant memoirs of recent years (posthumously published). The definition in the form given above was introduced independently at about the same time by Professor Karl Menger of Vienna and developed in a sequence of shorter papers characterized by their elegance and generality. The essentials of the dimensionality theory, which has by now attained a considerable perfection through the recent writings of Menger, Hurewicz, Alexandroff and others, have been developed with admirable clarity and completeness in a recently published book by Professor Menger. Let us review briefly some of the notable results of the theory as it is presented by the author.

A subset of a given space may itself be regarded as a space. If each of the subsets $M_1, M_2, \cdots$, is n-dimensional, is the set $\sum M_i$ n-dimensional? The answer is yes provided the M_i are closed. That the situation is entirely different in the general case is seen from Urysohn's remarkable theorem that an n-dimensional set is the sum of $n+1$, but not of fewer than $n+1$, 0-dimensional sets. On account of its obvious importance, the summation theorem for closed sets is developed by the author at the very outset. There is included an extremely elegant proof by Hurewicz which yields at the same time the Urysohn theorem. The polish of the modern methods is strikingly revealed when this proof is compared with the original proof of Urysohn.

In 1911 Lebesgue pointed out the following property of euclidean n-space: if a bounded region is covered by a finite number of closed sets of sufficiently small diameter, there must exist points which are common to $n+1$ of the covering sets. This remarkable property of ordinary space holds with suitable modifications for abstract n-dimensional spaces. Certain striking generalizations, together with a sort of converse which states that covering sets with intersection properties analagous to those of the (closed) n-cells of an n-complex, can always be chosen,—are among the author's own achievements. For reasons which we shall later indicate, the importance of this group of theorems can hardly be overestimated.

A significant feature of the theory as it stands at present is the relative unimportance of the assumption of compactness. The extension of the greater part of the theory to the more general separable metric spaces,—made possible by recent improvements in methods, furnishes further justification for the modern definition of dimensionality. The permanence, in this sense, of the theorems of dimensionality is analogous, as the author points out, to the permanence of the laws of arithmetic on the extension of a number domain.

There are, however, certain interesting exceptions. If a space R is n-dimensional, it must certainly be n-dimensional at certain of its points. Now if R^n represents the totality of such points, what is the dimension of R^n? If we assume that R is compact, the answer is that R^n is n-dimensional *at each of its points*. Quite different is the situation if R is not com-

pact. In fact we have actually an example, due to Sierpiński, of a one-dimensional space which is one-dimensional at only a denumerable set of points, and these points constitute a zero-dimensional set by the summation theorem. The author has succeeded in demonstrating that R^n as a space can not be of fewer than $n-1$ dimensions. Whether or not R^n may be less than $(n-1)$-dimensional at certain of its points remains one of the important unsolved problems in the structure analysis of separable spaces.

The cartesian spaces S_n are of course n-dimensional in the sense under consideration; were this not the case, the definition of dimensionality would fail to satisfy the most fundamental requirement. Since dimensionality is clearly a topological invariant, we have here a proof of Brouwer's "classical" theorem concerning the topological non-equivalence of S_p and S_q where $p \neq q$. It is this theorem which imparts validity to the definition of n-dimensional complex in the sense of combinatorial analysis situs.

We see then that one aspect of the problem of determining completely the relations between abstract spaces and number spaces of a given dimensionality is solved. Another aspect, concerning which there is little known as yet, we may characterize as the problem of the introduction of coordinate systems into point-set theoretical geometry. This problem is of extreme importance, since it is just here that a fusion between combinatorial analysis situs and point-set theoretical analysis situs is beginning to take place.

The few facts in this connection which are known are highly interesting and suggestive. We have for example the fact already referred to concerning the complex-like structure of an arbitrary space of finite dimensionality. There is also the recent result due to the author, and destined, it seems, to be of considerable importance, that every n-dimensional space R is homeomorphic to a subspace of a cartesian S_{2n+1}. That this theorem holds when R is a complex is obvious from the remark that two S_n's in S_{2n+1} fail in general to intersect. It is this fact together with the complex-like structure of R, which furnishes a guiding principle for a proof of the general theorem. The question as to whether or not S_{2n+1} is the space of lowest dimensionality in which R can be immersed, is still one of the interesting unsolved problems.

We mention finally the theorem of Alexandroff that if R is a closed n-dimensional space immersed in a euclidean space, there exists an n-dimensional approximating complex into which R can be carried by means of a singular continuous deformation in which each point moves an arbitrarily small distance; such an approximating complex can not be of fewer than n dimensions. Thus the permanence of dimensionality under a distortion depends on the magnitude of the distortion. It is essentially this fact which constitutes the germ of Brouwer's original proof of the "classical" invariance theorem.

It is impossible to compare Menger's book with the original sources without admiring the thoroughness and elegance with which he has embraced the essentials of the dimensionality theory as it stands today into a single volume of some three hundred pages. Nor can it be said that this has necessitated undue condensation. One may complain perhaps that the historical notes at the end of each section, while quite complete, avoid in

their conciseness certain questions of emphasis. But as for the subject matter, there is no feeling of compression; on the contrary the author frequently pauses to suggest problems, explain difficulties and guiding principles, and emphasize values. For some theorems there are developed several distinct proofs, each bringing to light new aspects of the theory and new points of view. The essentials of the point set theory of separable spaces are developed at an early stage so that one may read the entire volume without outside reference. The material is well arranged and the printing is excellent, with exceptionally few errors. The book constitutes, in short, a notable presentation of an important chapter in modern mathematics.

P. A. SMITH

TONELLI ON TRIGONOMETRIC SERIES

Serie Trigonometriche. By Leonida Tonelli. Bologna, Nicola Zanichelli, 1928. viii+523 pp.

The extent of the existing literature on the theory of trigonometric series is tremendous and keeps increasing very rapidly. During the quarter of a century after 1900 the theory has made remarkable progress and it would not be an exaggeration to say that now it is of equal and fundamental importance for all branches of the modern mathematics including the theory of numbers on the one hand, and mathematical physics on the other. In spite of, or perhaps because of this, there is practically no place in the literature where an adequate account of the theory is given, except for the second volume of Hobson's *Theory of Functions* and two excellent but short reports by M. Plancherel (L'Enseignement Mathématique, vol. 24 (1925), and by E. Hilb and M. Riesz (Encyklopädie der mathematischen Wissenschaften, vol. II, 3_2, 1924).

Under such circumstances the publication, by a mathematician of Tonelli's rank, of a large volume devoted exclusively to the theory of trigonometric series must be considered as a significant event, even if it does not represent a step toward the solution of the difficult problem of creating an all-inclusive treatise on trigonometric series.

The work under review originated as a course of lectures on trigonometric series delivered at the University of Bologna in 1924–1925 with a view "to expounding in a systematic manner the classical results, together with more recent investigations on these series." The author found it more convenient to abandon the usual order of treatment of trigonometric series. The book begins with a study of general trigonometric series, in order "to reveal at once the properties which are common to all such series," and subsequently passes to the discussion of the special properties of Fourier series. An advantage of such a treatment lies, according to the author's opinion, in the fact that "the theory of the general trigonometric series can be presented in a form essentially elementary in character, while the theory of Fourier series, for its complete development, requires speculations of a more advanced nature." No space is given to the "beautiful investiga-

tions of Denjoy concerning the computation of the coefficients of an everywhere convergent trigonometric series, nor to the important theory of quasi-periodic functions" of H. Bohr. The author expects to devote a separate volume to the applications of the theory to geometry and mathematical physics.

The topics which are most completely treated in the monograph under review are: the absolute convergence of the general trigonometric series, Riemann's method of summation, the representation of functions by trigonometric polynomials and, above all, double Fourier series.

A short historical sketch is followed by Chapter I (General trigonometric series; §1, necessary conditions of convergence; §2, sufficient conditions; §3, necessary and sufficient conditions). §1 contains a detailed analysis of the Cantor theorem and of the problem of absolute convergence of trigonometric series. The reader will find here some important theorems (Steinhaus, Denjoy-Lusin) which are not so well known as they should be. §2 treats primarily of the uniform convergence and of the convergence almost everywhere on $(0, 2\pi)$. The principal tool in treating the former of these two problems is of course the classical Abel transformation which figures here under the name of Brunacci-Abel transformation, with a reference to the *Corso di Matematica Sublime* by V. Brunacci (vol. 1, Firenze, 1804). As to the latter problem, we find here an important theorem due to Kolmogoroff-Seliverstoff and to Plessner: If $\sum(a_n^2+b_n^2)\log n$ converges, then the corresponding trigonometric series converges almost everywhere. It should be observed that the method of proof used is hardly of an "elementary" nature; on the one hand there is used an unproved lemma (p. 65) in the general theory of infinite series, which, so far as the reviewer is aware, is not stated in the standard treatises, like those of Bromwich or Knopp; on the other hand a reference is made to the Riesz-Fischer theorem, which is proved considerably later. The last §3 of Chapter I is devoted to the fundamental theorem of Riemann.

Chapter II (Representation of functions by means of trigonometric series) contains three parts: §1, conditions for the representation; §2, uniqueness of the expansion in a trigonometric series; §3, Fourier series. §§1 and 2 contain an interesting treatment of Riemann's method of summation and Cantor's uniqueness theorem, while in §3 the definition of Fourier series, together with some elementary examples, is given and the theorem of Du Bois-Reymond-Lebesgue is proved.

Chapter III (Approximate representation by means of trigonometric polynomials) contains four parts: §1, trigonometric interpolation; §2, method of least squares; §3, trigonometric polynomials of Fejér; §4, method of approximation of Tchebychev.

Chapter IV (Fourier successions) is subdivided into three parts: §1, Fourier sequences and series; §2, necessary conditions for Fourier successions; §3, sufficient conditions for Fourier successions. In §1 a short discussion of the uniqueness of determination of a function by means of the sequence of its Fourier coefficients is given. §2 contains the Riemann-Lebesgue theorem, an estimate of the order of magnitude of the Fourier coefficients of a function under various conditions, the Parseval identity,

the Young-Hausdorff theorem, the approximation theorem of Bernstein, the theorem of Szász and, finally, a discussion of convergence of the series $\sum a_n/n, \sum b_n/n, \sum a_n, \sum b_n$. The proof of the Young-Hausdorff theorem proceeds along the lines of the proof of F. Riesz with one deviation, however, which makes the proof more difficult; in establishing the existence of a minimum of a certain integral the reader is referred, not to the classical paper by F. Riesz in Mathematische Annalen, vol. 69, but to a general theorem of the calculus of variations as given in the second volume of the author's *Fondamenti di Calcolo delle Variazioni*. §3 is devoted to the proof of the Riesz-Fischer theorem and its corollaries.

Chapter V (Convergence of Fourier series) consists of six parts: §1, absolute convergence; §2, simple convergence; §3, uniform convergence; §4, partial convergence; §5, conjugate series; §6, degree of approximation. In §1 the theorem of Bernstein and some of the author's results are proved. §2 contains a discussion of the Dirichlet integral and of the classical criteria of convergence. One of these criteria (that of Young) is not mentioned and a new special criterion (due to the author) is added, namely, a "criterion of convergence at singular isolated points." The usefulness of this important paragraph would have been considerably increased if the author had given space to a comparative analysis of various criteria and showed examples of their applications, his own criterion included. In §4 we find an interesting theorem of Kolmogoroff: If $S_n(x)$ is the nth partial summation of the Fourier series of a function $f(x)$, then the sequence $\{S_{n_k}(x)\}$ converges to $f(x)$ almost everywhere, provided there is a constant $\lambda>1$, such that $n_{k+1}/n_k>\lambda$. The treatment of the conjugate series as given in §5 is rather short and hardly gives a right idea of the present situation of this question whose importance becomes more and more obvious.

Chapter VI (Operations on Fourier series) consists of three parts: §1, addition and multiplication; §2, integration; §3, differentiation. A remarkable theorem of Lusin, given in §2, deserves to be mentioned separately: If a trigonometric (not necessarily a Fourier) series converges at all points of (a, b) to a sum $S(x)$ which is integrable on (a, b), then the series can be integrated term by term over any subinterval which is entirely interior to (a, b).

Chapter VII (Singularities of the Fourier series: §1, singularities of the Fourier series of continuous functions; §2, singularities of the Fourier series of discontinuous functions) gives a discussion of singularities of Du Bois-Reymond and Lebesgue, and an account of the Gibbs phenomenon.

Chapter VIII (Classical integrals: §1, Poisson integral; §2, Fourier integral). §1 contains an exposition of classical results concerning the Poisson integral, including some of the results of Fatou. The treatment of the Fourier integral in §2 proceeds along the lines of Jordan-Pringsheim and of the Acta Mathematica paper by Hahn. The end of this paragraph is devoted to a discussion of the Sommerfeld integral.

Chapter IX (Double Fourier series: §1, preliminaries; §2, convergence of the double Fourier series; §3, summation by rows and columns; §4, Fejér trigonometric polynomials in two variables; §5, supplements). This concluding chapter is perhaps one of the most interesting in the volume. It is

based to great extent upon the author's own investigations and contains many results which reach farther than those known before. Considerable progress and simplification were made possible by introducing a new notion of a function of bounded variation in two variables, which contains most of those given before (Hardy, Vitali-Lebesgue-Fréchet, Arzelà): A function $f(x, y)[a \leqq x \leqq b,\ a \leqq y \leqq b]$ is of bounded variation in (x, y) if (i) $f(x, y)$ is of bounded variation in y for almost all values of x and of bounded variation in x for almost all values of y; (ii) the corresponding total variations are integrable as functions of x or of y respectively.

The selection of material in a course of lectures which is not to be considered as a treatise is to great extent a matter of taste. It is to be regretted, however, that such important topics as the theory of Fourier transforms, the general Cesàro summability of Fourier series (only C_1 summability is given consideration in the book), the relationship between the Fourier and power series, are not discussed in the volume under review. It is hardly possible to obtain a correct idea of the recent development of the theory of trigonometric series and integrals without touching these topics. But even if we concentrate our attention on the topics treated in the book, we might find several facts and names which perhaps deserve somewhat more than to be merely mentioned in a few footnotes. Such are, in the reviewer's opinion, important investigations of Hardy-Littlewood on the theory of Fourier constants (=Fourier successions), on the so-called strong summability of the Fourier series by the same, the theory of convergence factors by various writers, some investigations by Zygmund, Titchmarsh, Wiener, etc. There are also some names which should be referred to, at least in the footnotes, for instance, the name of Carleman, in connection with the discussion of the exponent of convergence of a Fourier succession of a continuous function, of the strong summability of Fourier series, etc.

Despite these criticisms we are convinced that *Serie Trigonometriche* by Tonelli will be of great value to anyone seriously engaged in studying the immense field of trigonometric series, and we are looking forward to the pleasure of reading the promised volume on the applications of trigonometric series.

J. D. TAMARKIN

BOREL ON DIVERGENT SERIES

Leçons sur les Séries Divergentes. Par Émile Borel. Deuxième édition, revue et entièrement remaniée avec le concours de Georges Bouligand. Paris, Gauthier-Villars, 1928. vi+260 pp.

Those who have followed the progress made in the theory of divergent series during the past quarter of a century are well aware of the important rôle which the first edition of Professor Borel's book has played in stimulating the development of this theory. This was due, not only to the importance of the results contained in it, but also to the elegance of the exposition and the many suggestive lines of thought that were indicated.

Throughout the period referred to Professor Borel's book has remained a classic in its field, but in view of the rapid growth of the theory it was inevitable that certain portions of the discussion should seem somewhat lacking in completeness after a lapse of twenty-five years. It was therefore quite appropriate that a second edition should be issued, bringing up to date the sections where later research had thrown fuller light on the questions involved, and giving a succinct resumé of the many important developments in various branches of the theory that have been made since the date of the first edition.

The account of recent work is for the most part contained in an additional chapter of some thirty-five pages at the close of the volume. Considering the extent of the literature involved,* it was obviously somewhat of a task to give a coherent account of it in such brief space. The difficulties of the undertaking have been overcome in admirable fashion, and while necessarily many important studies had to be dismissed with merely a reference, the chapter on the whole gives an excellent bird's-eye view of the research material with which it deals. By selection of certain central lines of thought that have characterized the research in question, it has been possible to unify many separate studies and thus make a considerable gain in the brevity and directness of their treatment.

Theoretical discussions of divergent series can be broadly classified in groups in accordance with their relationship to two fundamental methods of procedure. The methods in question may be designated briefly as follows: (1) summation by means; (2) summation by convergence factors. In many studies one or the other of these methods is used exclusively; others involve both of them, and some deal explicitly with various relationships between the two methods. In the first edition of the *Leçons* summation by means played the dominating rôle, and naturally the same thing is

* In Smail's *History and Synopsis of the Theory of Summable Infinite Processes*, which deals with work through the year 1922, there are listed more than 250 publications since the appearance of the first edition of Borel's book. The synoptic account of these books and memoirs, in most cases quite brief, occupies more than 150 pages.

true of that part of the present edition which constitutes a revision of the first edition. However, in Chapter III, which deals with the general theory of summable series, attention is called at the beginning to these two methods. This is done in connection with an illustrative discussion of the application of the methods of divergent series to the study of Fourier's series. No better introduction to the two points of view could have been selected, since the application to Fourier's series furnishes a simple and illuminating example of the efficiency of the two methods and the interplay between them.

In the added chapter of the present work, Chapter VI, a condensed account is given of a variety of studies which center about the method of convergence factors. As some of the most fundamental work of recent years based on the method of means has already been treated in Chapter III, the latter method is more briefly dealt with here. In the field of applications of divergent series, the account is limited in the main to work in connection with Dirichlet's series, factorial series, and quasi-analytic functions.

Of the chapters that occur in the first edition, Chapter V, dealing with developments in series of polynomials, is carried over unchanged. The reason for this is well stated in a footnote by Professor Bouligand, who remarks that the chapter was not merely of a didactic character, but exhibited the contact of ideas of two great mathematicians.* Of the remaining chapters, those dealing with asymptotic series, continued fractions and the theory of Stieltjes, and analytic extension by the method of Borel summability cover practically the same ground as before. Many sections have been revised and certain amplifications and modifications suggested by later researches have been made, but the subject matter is substantially the same.

The chapter on the theory of summable series contains much new material, being about twice the size of the corresponding chapter in the first edition. Among the new topics introduced, in addition to the discussion of Fourier's series referred to above, may be mentioned a more complete discussion of the means of Cesàro and Hölder, including a proof of their equivalence, and some work of Sannia and Le Roy connected with Borel summability.

The final chapter of the book is followed by an interesting note by Professor Bouligand, demonstrating the impossibility of securing analytic extension by the use of means of certain general types. This note is followed by a list of results culled from various memoirs on divergent series and integrals, stated in the form of exercises.

C. N. MOORE

* Mittag-Leffler and Borel.

KÖNIG AND KRAFFT'S ELLIPTIC FUNCTIONS

Elliptische Funktionen. By R. König and M. Krafft. Leipzig, Walter de Gruyter and Co., 1928. 263 pp.

Many are the methods of approach to the study of elliptic functions. Since Abel's classic memoir (1826), an outgrowth of his researches on the lemniscate definite integral, much attention has been given to the development of the theory of algebraic and of abelian functions and of the chapter in this theory which has to do with the subject of this book. Some, as Riemann (1857) and Weierstrass (1875–6) followed first the lead of Abel in the 1826 paper (anticipated by Gauss as early as 1798) but later redeveloped the theory using the algebraic differential equation of the first order as a cornerstone on which to build. Others, as Jacobi, (1829 and 1834), and Eisenstein, starting with properties of multiply periodic functions, have developed the theory of elliptic functions as an elegant chapter in the theory of functions. The geometric properties and applications interested Riemann, Clebsch, (1864), and Brill and Noether, (1874); while the latest and indeed the least traveled avenue of approach, viz., the number-theoretic, called also arithmetic, was employed by Dedekind and Weber, (1879), by Kronecker, (1881), and by Hensel and Landsberg, (1902).

This book by König and Krafft gives the first detailed exposition of the arithmetic theory as applied specifically to functions having the three fundamental properties which characterize elliptic functions, viz., (1) they form a linear manifold, (2) they form an algebraic field, (3) they are of genus one. The aim has been "not to set up theorems and formulas with lemmas and computations constructed for the purpose but to develop the results organically and genetically" from these three properties of which the first is the simplest and at the same time the most general. Thus it is evident that the scope of the book is considerably more than the development of the arithmetic theory of functions of genus one. As the introduction tells us: The reader becomes familiar with Riemann's doctrine, which lies at the bottom of the whole plan of the book, a doctrine which not only supplies a Riemann surface on which the function is single-valued, but which also enables the reader to understand the classification of elliptic functions as a member of a great organism, Riemann function-pairs as an instance of Riemann function-systems; moreover, according to Riemann and F. Klein, the fundamental law which rules the organism is a law of duality by which every theorem on functions can be restated for differentials. The reviewer thinks the book justified by their claim that "though the material is not new, the contribution is in the painstaking methodical simplifications made even in the things well known." It is a contribution which is welcome. The book can be read with pleasure by one who is familiar with the theory of elliptic functions in any of its phases or ramifications, and it can be used quite satisfactorily as a text with a class which knows the elements of function theory though not without considerable amplification of the

material by the instructor. In particular the great array of notational symbols in the book makes the reading more difficult than the mathematical theory should require. If it could have been presented without this piling of symbol upon symbol it would have been better.

After reading the first three chapters a reader can turn to the general theory of algebraic functions as given in Hensel and Landsberg's treatise and in Chapter 24 of the third volume of Weber's *Lehrbuch der Algebra* and get the extension to an n-valued algebraic function with considerable ease. One should not assume that because the elliptic case is a special instance of the general case it is not worth while to study it in as detailed a fashion as is done here. As a matter of fact the translation of the notions from general (p arbitrary) to special ($p=1$) are far from obvious while the wide applications of the elliptic case make the special case important per se.

The list of chapter headings gives an indication of the scope of the book:

1. Grundlagen.
2. Arithmetischer Teil.
3. Die Elementarfunktionen.
4. Elementarfunktionen und Elementardifferentiale von zwei Veränderlichen.
5. Die Elliptischen Integrale.
6. Funktionen zweiter und dritter Art.
7. Das Abel'sche Theorem.
8. Konforme Abbildung durch elliptische Funktionen und birationale Transformationen.
9. Die elliptischen Funktionen und Integrale im Periodenparallelogramm.
10. Die Funktionen des Ringes.

In the first chapter are given fundamental definitions and notations. An elliptic algebraic function is single-valued and regular save for a finite number of poles on a two sheeted Riemann surface with four branch points. It is a special case of a class (K) of (Riemann) function-pairs on two sheets on which the branch cuts have been made but not joined. All properties of elliptic functions as fast as developed are discussed with respect to elliptic differentials.

Chapter 2 gives the arithmetic part. Using the methods of number theory we can find a basis for the entire class; an ideal basis for the ideal of the divisor, Q; a multiple basis for the divisor, Q, at all finite points; all functions which are multiples save possibly at P_∞ of a given divisor; a normal basis for such functions; and finally a basis which is normal at P_∞.

Chapter 3 treats of the characterization of elliptic functions and differentials. An elementary function with a fixed pole of arbitrary order is the element in terms of which any elliptic function is expressible. This elementary function plays a rôle similar to the fraction $1/(x-a)$ in the decomposition of rational functions into partial fractions. A corresponding elementary differential is defined. The method is extended to functions of two complex variables in Chapter 4, while in Chapters 5 and 6 elliptic integrals on the Riemann surface which has been made simply connected

by $2p$ period cuts are constructed in terms of integrals of the elementary differentials of two complex variables. Here the Weierstrass, Klein, and Prym prime functions are defined, again in terms of the elementary functions.

Chapter 7 gives an elegant exposition of the theory underlying Abel's theorem for elliptic integrals of the three kinds. In these seven chapters the functions have been studied on the two-sheeted Riemann surface.

In Chapter 9 the functions are studied on the period parallelogram which in its turn in Chapter 10 is mapped conformally on a ring. This follows an excellent discussion of conformal mapping by elliptic functions and birational transformations in Chapter 8. The arithmetic method is abandoned now for the more usual methods of analysis. The authors tell us that the shriveling of the theory due to the fact that now functions and differentials become identical gives greater simplicity but with loss in beauty and richness of structure. The Fourier developments of elliptic functions are obtained directly from known properties of integrals on the ring and may then be carried back to the period parallelogram. The simplicity of this exposition will appeal to those who know how Fricke obtained the same results by the application of functional equations.

The work of printer and editor have been well done. The text is almost entirely free from typographical errors.

MAYME I LOGSDON

SHORTER NOTICES

Abriss der Logistik. By Rudolf Carnap. Wien, J. Springer, 1929. iv+114 pp.

The first part of this "outline" will be of interest primarily to those who desire a brief introduction to modern symbolic logic and the foundations of mathematics. Dr. Carnap, following, for the most part, the usages, development and analyses of the *Principia Mathematica*, expresses in the Peanesque notation the more important concepts and propositions of propositional logic, the logic of classes, the logic of relations, cardinal arithmetic, series, progressions, continua, etc. New and difficult concepts are frequently illuminated by simple illustrations. A considerable number of new symbols are introduced, and a definite formulation of the principle of substitution, (inexplicably omitted from the *Principia Mathematica*), is provided.

In the second, more important and original part, in terms of a few primitive ideas for each discipline, some of the fundamental concepts and propositions of such diverse studies as the theory of aggregates, geometrical topics, projective geometry, modern physics, causality, epistemology, etc., are represented. This section (together with the appendix, where a number of suggestions are offered for alternative approaches to the same subjects) marks a considerable extension of logistic; though, as the novel applications are not thoroughly exploited, and only obvious deductions and analyses are provided, there is no surety that the chosen primitives are always adequate for the given disciplines.

Logistic developments are never final. They are merely intermediate steps in the derivation of special sciences from logic. As all ideas are expressible in terms of class concepts, relations and associated notions, the "axioms" (Carnap's term) of all exact, logisticized sciences should be expressible as propositions which are completely derivable from the concepts and propositions of logic. Such a procedure will immediately reveal the strength and weakness of the familiar thesis (which Dr. Carnap accepts), that mathematics is a branch of logic. How much caste that thesis will retain when it is realized that, in the same sense in which mathematics is a part of logic, physics, epistemology, biology, grammar, etc. are also parts of it, is a matter for speculation. That realization, however, will bring home the as yet unrecognized necessity for a rediscussion of the nature of necessary truths, inasmuch as there does seem to be a sense in which the propositions of logic are necessary apart from all systems, in contrast to those of mathematics and other essentially systemic sciences.

The first part of this work is an excellent pocket *Principia Mathematica*; the second part should prove of value to those who are interested in extensions of logistic; as a whole it should help provide answers to questions regarding the significance of the logistic and logico-mathematical movement.

PAUL WEISS

The Foundations of Euclidean Geometry. By Henry George Forder. xii+349 pp. Cambridge University Press, 1927.

Rarely is the object of a book so tersely expressed on the wrapper as in the case of the foundations of euclidean geometry. The publishers quote these words from the preface: "Although the euclidean geometry is the oldest of the sciences and has been studied critically for over two thousand years, it seems there is no text-book which gives a connected and rigorous account of that doctrine in the light of modern investigations. It is hoped that this book will fill the gap."

There may be a difference of opinion on the question of the completeness of the gap filling, but Mr. Forder has written an interesting book and one which should be welcomed by those interested in seeing the rigor of modern analysis achieved in geometrical studies.

Being anxious to achieve exactness, the author intersperses his definitions and axioms with copious notes explaining the advantages of particular statements. Thus, for example, he uses forty pages to establish his concepts of classes, relations, linear order, non archimedean systems, etc.

With this background he proceeds to set up axioms of order, which he points out were "almost completely ignored by Euclid, but which are fundamental for this work."

Starting with point and order as undefined entities definitions are made of line, plane, and space followed by development of fundamental properties of these.

"We have now laid the foundations of geometry. Our Axioms OI–VIII (order axioms) together with a continuity axiom and a Euclidean parallel axiom enable us to erect a complete Euclidean Geometry. Also—they enable us to introduce algebraic methods."

Following the introduction of angles and their properties, the question of congruence is considered. Here are first introduced the congruence axioms. With the introduction of circle axioms it is shown that the congruence axioms may be greatly weakened.

The introduction of the parallel axiom is deferred to the sixth chapter and is then stated in not fewer than eight different forms. In this chapter is also considered the relations between projective and euclidean spaces, ideal elements are defined and the addition of these to euclidean elements is discussed.

The Desargues and Pappus theorems come in for lengthy discussion from various points of view, while constructions are given a chapter to themselves with special attention to the Poncelet-Steiner and Mascheroni theorems and to the Gaussian theory of the regular polygon.

After chapters on analysis situs of polygons, areas of polygons and volumes of tetrahedra the author says: "We now reach the climax of our investigation. So far we have not assumed that our lines are continua. We shall now secure this property by assuming a continuity axiom." He completes his argument by showing that all euclidean geometry may be logically built upon order axioms, a very restricted parallel axiom and a weakened continuity axiom.

The last two chapters are quite distinct, almost in the nature of an appendix "designed to illustrate one or two interesting points" such as congruences as the sole undefined relation between points, non-euclidean areas, etc.

Mr. Forder has undertaken a difficult task and, while his work may be open to criticisms in some respects, he has made a readable book, the study of which by our teachers of euclidean geometry might go far toward eliminating the inaccuracy of concepts and illogical deductions from the minds of those supposedly engaged in teaching accurate, precise thinking. It is a worth while book which may well serve as a stimulus to others to write into our literature more exact presentations of our basic geometric concepts.

F. W. OWENS

Operational Methods in Mathematical Physics. By Harold Jeffreys. London, Cambridge University Press, 1927. vii+101 pp. Price 6 s. 6 d.

This valuable monograph is number 23 of the important series entitled Cambridge Tracts in Mathematics and Mathematical Physics. The author says in the preface: "My own reason for writing the present work is mainly that I have found Heaviside's methods useful in papers already published, and shall probably do so again soon, and think that an accessible account of them may be equally useful to others." As a matter of fact he has filled admirably a gap of some thirty years standing in the literature pertaining to the solution of the differential equations of physics, since Heaviside's own work is not systematically arranged and in places its meaning is rather obscure. Jeffreys also affirms that "· · · it is certain that in a very large class of cases the operational method will give the answer in a page when ordinary methods take five pages, and also that it gives the correct answer when the ordinary methods, through human fallibility, are liable to give a wrong one."

A general idea of the scope of the text may be derived from the chapter headings, which are in order: Fundamental Notions, Complex Theory, Physical Applications: One Independent Variable, Wave Motion in One Dimension, Conduction of Heat in One Dimension, Problems with Spherical or Cylindrical Symmetry, Dispersion, and Bessel Functions. A wealth of illustrative material is compressed within these main divisions. Thus, for example, the third chapter deals with the following topics: (1) variation with time of the electric charge on the condenser plates for a circuit containing a voltaic cell, a condenser, and a coil possessing inductance and resistance; (2) the Wheatstone bridge method of determining inductance; (3) the seismograph, with special reference to the instruments of Galitzin and Milne-Shaw; (4) resonance for a simple pendulum; (5) motion of three particles attached to a stretched string; (6) radioactive disintegration of uranium; and (7) some dynamical applications.

The monograph will be interesting to mathematicians as well as to physicists because it brings out difficulties which require further investigation. For illustration, the first sentence on page 53 is: "A general proof that the results given by the operational method, when applied to the

vibrations of continuous systems, are actually correct, has not yet been constructed." Likewise the comments in section 8.8 are suggestive and illuminating. The volume closes with a note on the notation for the probability integral, with a page on interpretations of the principal operators, with a bibliography of papers in which operational methods are used, and with author and subject indexes.

H. S. UHLER

Nouvelles Tables de Log n! By F.-J. Duarte. Paris, Index Generalis, 1927. xxiv+136 pp.

This book contains in large clear type the common logarithms to 33 decimals of factorial n from $n=1$ to $n=3000$. A preface is given by Professeur M. R. de M. de Ballore, who emphasizes the importance and reliability of the work. The largest previous table by Degen, (1824), contained 18-place logarithms up to log 1200! The aim of Duarte was to allow adequate study of expressions constantly occurring in the theory of probability.

Incidentally the author points out errors in the logarithms of 829, 1087, 1409, 1900 as given by Wolfram and published in Vega's Thesaurus, also in Thoman's values for log 45! and log 55! and Degen's values for log 1093! and log 1180!.

In 1925 Duarte and Ballore published a similar twelve-figure table to $n=1000$.

The present table was constructed by adding successively the logarithms to 39 decimals of numbers from 1 to 3000. To control the table Stirling's formula was used to calculate independently to 36 places log $50k!$ for $k=1, 2, \cdots, 60$. The forcing of a digit in the 33d place is indicated by an asterisk.

If p is a prime beyond 1000, its logarithm may be calculated by means of the remarkable series $\log p=\frac{1}{2}\log(p-1)+\frac{1}{2}\log(p+1)+\Delta_0+\Delta_1+2.8\Delta_2+72.4\Delta_3/7+304.48\Delta_4/7+R_4$, where $4p\Delta_0=\log(p+1)-\log(p-1)$, $6p^2\Delta_k=\Delta_{k-1}$, $k=1, 2, 3, \cdots$, and $R_4<.0^{38}5$. The author used a similar series to compute $\log p$ when $17<p<1000$ by a proper choice of m such that $mp-1$ and $mp+1$ contain prime factors less than p. He accordingly started with 42-place logarithms of 2, 3, 5, 7, 11, 13, 17 and calculated all the other primes by this new method.

Three obvious errors in printing occur on pp. xiv, xvi, and xxii.

In calculating a logarithm the method of Flower is used and illustrated by the evaluation of $\log\pi$ and $500!/40!\times460!$ The necessary table for Mn, $n=1, 2, \cdots, 100$; and the brief table for $\log 1.0^s k$, $s=0, 1, \cdots, 16$; $k=1, 2, \cdots, 9$, constitute Tables II, III, respectively, the former running to 40 decimals.

The reviewer was pleased to find a use for this new table in computing $\log\pi$ from the formula

$$\pi^{35}E_{17}=2^{36}\times34!(1-1/3^{35}+1/5^{35}-1/7^{35}+\cdots)$$

correct to 33 places.

C. C. CAMP

The Symmetrical Optical System. By G. C. Steward. London, Cambridge University Press, 1928. viii+102 pp. Price 7 s. 6 d.

This monograph is number 25 of the important series known as Cambridge Tracts in Mathematics and Mathematical Physics. The elementary theory of the first chapter is followed by a discussion of the characteristic function of Hamilton, the "eikonal," and the aberration function. The way is thus prepared for the immediate consideration of coma, distortion, and geometrical aberrations of the first and higher orders. A chapter is next devoted to interesting presentations and proofs of the sine-condition, the cosine-conditions, Herschel's condition, and the very general optical cosine law. The fifth chapter is devoted to the preparation of formulas especially adapted to the computation of optical systems. However, no attempt is made to apply these results to numerical data which have arisen in the designing of actual optical apparatus. The two remaining chapters take into account the finite wave-length of light, and they deal respectively with the diffraction patterns associated with the symmetrical optical system and with three forms of aperture, namely, the annular aperture, the slit aperture, and the semi-circular aperture.

The usefulness of the tract is enhanced by the inclusion of lists of references at the ends of all the chapters. The figures or line diagrams are clear, well chosen, and sufficiently numerous, and the monograph will constitute undoubtedly a valuable reference source both on account of the uniformly clear, logical presentation of the material involved and because of the powerful analytical methods developed and employed.

H. S. UHLER

Théorie des Champs Gravifiques. (Mémorial des Sciences Mathématiques, No. 14.) By Th. de Donder. Paris, Gauthier-Villars, 1926. 60 pp.

This is the second of a sequence of pamphlets on the theory of relativity which de Donder is contributing to the Mémorial des Sciences Mathématiques (for a review of the first of these pamphlets see this Bulletin, vol. 32, p. 563).

In this pamphlet the author first derives the equations of a gravitational field in all possible generality. He then considers the more important special cases, such as those fields due to "ordinary" gravitation associated with the presence of material bodies, those electromagnetic fields due to the presence of charged bodies, and those due to the simultaneous presence of matter and electric charges. As in the other pamphlets of the Mémorial series, we have here a minimum of formal derivation with many references to the literature, the style suggesting that of the Encyklopädie. As a result we have a very good bird's eye view of a very wide domain.

C. N. REYNOLDS, JR.

NOTES

In addition to those on the editorial staff of the Bulletin, the following persons have assisted the editors, either by refereeing papers or by advising concerning papers offered for publication in the present volume: A. A. Albert, Harry Bateman, E. T. Bell, B. A. Bernstein, H. F. Blichfeldt, Henry Blumberg, R. D. Carmichael, A. B. Coble, J. L. Coolidge, H. T. Davis, L. E. Dickson, L. L. Dines, G. C. Evans, Peter Field, T. C. Fry, Raymond Garver, Lois W. Griffiths, G. H. Hardy, T. R. Hollcroft, E. V. Huntington, Edward Kasner, J. R. Kline, E. P. Lane, C. H. Langford, C. G. Latimer, W. A. Manning, Marston Morse, F. D. Murnaghan, J. H. Neelley, James Pierpont, J. F. Ritt, J. A. Shohat, Virgil Snyder, E. B. Stouffer, J. D. Tamarkin, H. S. Vandiver, Warren Weaver, J. H. M. Wedderburn, G. T. Whyburn, W. M. Whyburn, D. V. Widder, R. L. Wilder, A. R. Williams, W. A. Wilson, R. M. Winger, B. C. Wong, F. E. Wood, J. W. Young. The editors desire publicly to recognize this service.

On account of the death of Dr. J. W. L. Glaisher, editor of the Quarterly Journal of Mathematics and the Messenger of Mathematics, the former journal will be discontinued, with the completion of its fiftieth volume. Professor G. H. Hardy has consented to edit the current volume (volume 58) of the Messenger of Mathematics, and it is hoped that this journal can be continued.

The first Congress of Roumanian Mathematicians was held at the University of Cluj, May 9-12, 1929. The attendance included one hundred fifty persons, and fifty communications were read.

Professor Florian Cajori, of the University of California, has been elected a vice-president of the Comité International d'Histoire des Sciences, which met at Paris, May 24, 1929. The official organ of this society is Archeion, edited by A. Mieli.

The Swiss Mathematical Society has established a foundation for the advancement of the mathematical sciences in Switzerland. The income of this foundation is to be used for the support of the Commentarii Mathematici Helvetici and other mathematical publications.

At the tenth annual meeting of the American Geophysical Union, Dr. William Bowie was elected chairman of the Union, and Mr. W. D. Lambert chairman of the section of geodesy.

The Ernst Abbe Memorial Prize for mathematics and physics for 1928 has been awarded to Professor Alexander Meissner, of Berlin, "dem Erfinder des Röhrengenerators."

The committee on astronautics of the Astronomical Society of France

has awarded its first annual international prize to Hermann Oberth, of the College of Medinsch, Roumania.

The New York Academy of Sciences offers its A. Cressy Morrison prize "No. I" for 1930 for a paper on solar and stellar energy. Competition is open to all.

On the occasion of its centenary celebration, the Stuttgart Technical School has made Professors A. Stodola of Zurich, and S. Finsterwalder, of Munich "Ehrenbürger," and has conferred an honorary doctorate on Professor E. Trefftz, of Dresden. The Zurich Technical School has conferred honorary degrees on Professors F. Emde and R. Grammel, of Stuttgart.

Professor O. Chisini, of the University of Milan, has been elected correspondent of the Reale Istituto Lombardo.

On the occasion of its seventh centenary celebration, the University of Toulouse conferred an honorary doctorate on Professor Tullio Levi-Civita, of the University of Rome. Professor Levi-Civita has also recently received an honorary doctorate in engineering from the Technical School of Aachen, for his work in mechanics and especially in hydrodynamics.

The University of Pisa has established a foundation for the advancement of mathematics, in honor of the late Professor L. Bianchi, of that university.

The University of Durham has conferred an honory doctorate on Sir Frank Dyson, astronomer royal.

Assistant Professor E. C. Bower, of the department of astronomy and mathematics at Ohio Wesleyan University, has been appointed Martin Kellogg fellow at the Lick Observatory of the University of California, with academic residence at Berkeley.

Dr. Deborah M. Hickey has been awarded an Alice Freeman Palmer fellowship at Wellesley College.

Dr. Alexander Pogo, of the Yerkes Observatory, has been appointed fellow in the history of science by the Carnegie Institution of Washington. He will work at Harvard University.

Dr. E. Esclangon, director of the Strasbourg Observatory, has been appointed director of the Paris Observatory.

Professor F. Gonseth, of the University of Bern, has been called to the professorship of higher mathematics at the Zurich Technical School, as successor to Professor J. Franel, who has retired.

Dr. Hermann Müntz, of the Institut Einstein in Berlin, has been appointed professor of mathematics at the University of Leningrad.

Mr. N. F. Mott has been appointed lecturer in mathematical physics at the University of Manchester.

At Ohio State University, Professor Alfred Landé, of the University of Tübingen, has been appointed visiting professor of theoretical physics for the autumn and winter quarters of 1929-30. Dr. L. H. Thomas, of Trinity College, Cambridge, has been appointed visiting assistant professor of theoretical physics for the autumn, winter, and spring quarters.

Dr. W L. Ayres has been appointed to an assistant professorship at the University of Michigan.

Professor D. P. Bartlett, of the Massachusetts Institute of Technology, has retired.

Professor R. D. Carmichael has been appointed administrative head of the department of mathematics at the University of Illinois, as successor to Professor E. J. Townsend, who has retired.

Professor W. L. Crum, of Stanford University, has been appointed professor of economics at Harvard.

Assistant Professor P. D. Edwards, of Ball Teachers College, has been promoted to an associate professorship.

Mr. W. I. Foster, of Rochester Junior College, has been appointed assistant professor of mathematics at Northern Montana School.

Dr. B. F. Kimball, of Cornell University, has been appointed to an assistant professorship at the University of New Hampshire.

Assistant Professor E. E. Libman, of the University of Illinois, has resigned to accept a position in the marine and air craft engineering department of the General Electric Company.

Associate Professor T. A. Pierce, of the University of Nebraska, has been promoted to a professorship.

Dr. Hillel Poritsky, has accepted a position with the General Electric Company.

Assistant Professor S. A. Schelkunoff, of the State College of Washington, has resigned to accept a position in the Bell Telephone Laboratories, New York City.

Mr. C. K. Sherer, of the University of Nebraska, has been appointed head of the department of mathematics at Texas Christian University.

Assistant Professor G. W. Smith, of the University of Kansas, has been promoted to an associate professorship.

Dr. Roxana H. Vivian has been appointed professor of mathematics at Hartwick College, Oneonta.

Mr. E. H. Wells, of Princeton University, has been appointed to an assistant professorship at the University of New Hampshire.

Dr. E. W. Woolard, of the George Washington University, has been promoted to an assistant professorship of mathematics.

Dr. H. A. Zinszer, professor of physics and acting professor of mathematics at Hanover College, has been appointed professor of physics and astronomy at the Kansas State Teachers College.

The following appointments to instructorships are announced:
Connecticut College, Miss Grace Shover;
Newark College of Engineering, Mr. J. H. Fithian;
New York University, Dr. D. A. Flanders, Mr. E. H. Johnson, Mr. A. S. Peters;
Ohio State University, Dr. P. M. Swingle;
Pennsylvania State College, Dr. Leo Zippin.

Mr. G. Coradi, known as a maker of mathematical instruments, especially integraphs and harmonic analyzers, died March 2, 1929, at the age of eighty-one.

Professor Emil Hilb, of the University of Wurzburg, died August 6, 1929.

Professor T. Lalesco, of the University of Bucarest, known for his work in integral equations, died June 15, 1929.

Professor F. Rudio, retired, of the Zurich Technical School, died, in June 1929, at the age of seventy-two.

Associate Professor Otto Schreier, of the University of Rostock, is dead.

Mr. G. R. Kaye, well known as a writer on the early history of Hindu mathematics, died July 1, 1929, at the age of sixty-two. He is best known for his elaborate study of the Bakhshālī Manuscript, appearing in the Archaeological Survey of India, volume 43, Calcutta, 1927. In all his writings, he tended to the belief that India owed her early mathematics only to the Greeks.

Professor F. C. Kent, of the Oregon State Agricultural College, died June 11, 1929. Professor Kent had been a member of this society since 1911.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

ABBOT (E. A.). Flächenland, eine Geschichte von den Dimensionen von einem Quadrat (Edwin A. Abbot); aus dem Originalwerk "Flatland" ausgewält und ins Deutsche übertragen von W. Bieck. Leipzig, Teubner, 1929. 40 pp.

BENNY (L. B.). Mathematics for students of technology. Junior course, first year. London, Oxford University Press, 1929. 158 pp.

BERZOLARI (L.). See ENCYKLOPÄDIE.

BIECK (W.). See ABBOT (E. A.).

BJERKNES (C. A.). Niels Henrik Abel. En skildring av hans liv og arbeide. Omarbeidet og forkortet utgave i anledning 100-årsdagen for Abels død. Ved V. Bjerknes. Oslo, Aschehoug, 1929. 155 pp.

BJERKNES (V.). See BJERKNES (C. A.).

BRESLICH (E. R.). Senior mathematics. Book 3. (University of Chicago Mathematical Series, School of Education Texts and Manuals.) Chicago, University of Chicago Press, 1929. 216 pp.

BUCHANAN (S.). Poetry and mathematics. New York, John Day, 1929. 197 pp.

COBLE (A. B.). Algebraic geometry and theta functions. (American Mathematical Society, Colloquium Publications, volume 10.) New York, American Mathematical Society, 1929. 8 + 282 pp.

COSTER (B.). Constructie van ruimteelementen. Groningen, Wolters, 1928. 63 pp.

CRATHORNE (A. R.). See RIETZ (H. L.).

DIJKSTERHUIS (E. J.). De Elementen van Euclides. Deel 1: De ontwikkeling der Grieksche wiskunde voor Euclides. Boek I der Elementen. Groningen, Noordhoff, 1929. 236 pp.

DURELL (C. V.) and SIDDONS (A. W.). Graph-book. An exercise book and text-book. London, Bell, 1929.

ECKHART (L.). Der vierdimensionale Raum. (Mathematisch-Physikalische Bibliothek.) Leipzig, Teubner, 1929. 54 pp.

ENCYKLOPÄDIE DER MATHEMATISCHEN WISSENSCHAFTEN. Band III, Teil 2, Heft 9: K. Rohn und L. Berzolari, Algebraische Raumkurven und abwickelbare Flächen. Leipzig, Teubner, 1928.

FIELDS (J. C.). See INTERNATIONAL MATHEMATICAL CONGRESS.

FORD (L. R.). Automorphic functions. New York, McGraw-Hill, 1929. 12 + 333 pp.

FORREST (S. N.). Calculus for technical students. London, Edward Arnold, 1929. 8 + 231 pp.

GALE (A. S.). See SMITH (P. F.).

GANS (R.). Vektoranalysis mit Anwendungen auf Physik und Technik. 6te, verbesserte Auflage. Leipzig, Teubner, 1929. 112 pp.

HILB (O.). Einführung in die Algebra. 2 Bände. Leipzig, Akademische Verlagsgesellschaft, 1929. 15 + 367 + 13 + 295 pp.

HUNTER (W.). Groundwork of calculus. London, University Tutorial Press, 1929. 6 + 220 pp.

HUYGENS (C.). Christiaan Huygens, 1629—14 April—1929, zijn geboortedag, 300 jaar geleden, herdacht. Amsterdam, H. J. Paris, 1929. 4 + 61 pp.

INTERNATIONAL MATHEMATICAL CONGRESS. Proceedings of the International Mathematical Congress held in Toronto, August 11-16, 1924. Edited by J. C. Fields. 2 volumes. Toronto, University of Toronto Press, 1928.

IVES (H. C.). Seven place natural trigonometrical functions. New York, Wiley, 1929. 222 pp.

KEYSER (C. J.). The pastures of wonder. New York, Columbia University Press, 1929. 12 + 208 pp.

LAINÉ (E.). Premières leçons de géométrie analytique et de géométrie vectorielle. Paris, Vuibert, 1929. 47 pp.

LONGLEY (W. R.). See SMITH (P. F.).

LOTZE (A.). Punkt- und Vektor-Rechnung. Berlin, de Gruyter, 1929. 192 pp.

NEELLEY (J. H.). See SMITH (P. F.).

PERRON (O.). Die Lehre von den Kettenbruchen. 2te verbesserte Auflage. Leipzig, Teubner, 1929. 12 + 524 pp.

RIETZ (H. L.) and CRATHORNE (A. R.). College algebra. 3d edition. New York, Holt, 1929. 16 + 273 pp.

ROHN (K.). See ENCYKLOPÄDIE.

SCHUH (F.). Lessen over de hoogere algebra. Tweede volledig herziene druk. Eerste deel. Groningen, Noordhoff, 1929. 584 pp.

SIDDONS (A. W.). See DURELL (C. V.).

SMITH (P. F.), GALE (A. S.), and NEELLEY (J. H.). New analytic geometry. Boston, Ginn, 1928. 10 + 326 pp.

SMITH (P. F.) and LONGLEY (W. R.). Mathematical tables and formulas. New York, Wiley, 1929. 5 + 66 pp.

VALENTINER (S.). Vektoranalysis. 4te, wenig veränderte Auflage. (Sammlung Göschen.) Berlin, de Gruyter, 1929. 136 pp.

VALIRON (G.). Familles normales et quasi-normales de fonctions méromorphes. (Mémorial des Sciences Mathématiques, No. 38.) Paris, Gauthier-Villars, 1929. 56 pp.

DE VRIES (H.). Beknopte differentiaal- en integraalrekening. Groningen, Noordhoff, 1929.

PART II. APPLIED MATHEMATICS

AMBRONN (R.). Elements of geophysics as applied to explorations for minerals, oil, and gas. Translated by M. C. Cobb. London, McGraw-Hill, 1928. 372 pp.

AMES (J. S.) and MURNAGHAN (F. D.). Theoretical mechanics. An introduction to mathematical physics. Boston, Ginn, 1929. 462 pp.

BAUER (G. N.). Mathematics preparatory to statistics and finance. New York, Macmillan, 1929. 8+337 pp.

BOUTARIC (A.). Les ondes Hertziennes et la télégraphie sans fil. Paris, Flammarion, 1929. 253 pp.

BOWLEY (A. L.). Eléments de statistique. Traduits sur la cinquième édition anglaise par L. Suret et G. Lutfalla. Paris, Girard, 1929. 12 + 616 pp.

BRICOUT (P.). Ondes et électrons. Paris, Armand Colin, 1929. 215 pp.

BUNDSCHU (F.). Angewandte Hydraulik. Berlin, Springer, 1929. 75 pp.

CABANNES (J.). La diffusion moléculaire de la lumière. Paris, Presses Universitaires de France, 1929. 8 + 326 pp.

CARTER (C. C.). See MONTEITH (C. N.).

COBB (M. C.). See AMBRONN (R.).

CONRADY (A. E.). Applied optics and optical design. Part 1. London, Oxford University Press, 1929. 9 + 518 pp.

CORRAL (J. I.). Relatividad eliptica. Parte III: Geometrizacion de la fisica en los universos positivos. (Republica de Cuba, Memorias Cientificas de la Direccion de Montes y Minas, tomo 2.) Habana, 1929. 132 pp.

COTTER (J. R.). See PRESTON (T.).

CROWTHER (J. A.). Ions, electrons and ionizing radiations. 5th edition. London, Arnold, 1929.

CUGNIN (L.). Structure de l'atome, tourbillon d'éther et pensées scientifiques indépendantes. Sceaux, Bry, 1929. 184 pp.

DEFRETIN (A.). Cours d'électricité industrielle. Tome 1. Paris, Hermann, 1929. 582 pp.

DETLEFS (H.). Darstellende Geometrie. Heft 3. Frankfurt a. M., Diesterweg, 1929.

DREW (C. E.). See DUNCAN (R. L.).

DULL (C. E.). Modern physics. London, Harrap, 1929. 8+778 pp.

DUNCAN (R. L.) and DREW (C. E.). Radio telegraphy and telephony. New York, Wiley, 1929. 10+950 pp.

EBERHARD (G.), KOHLSCHÜTTER (A.), und LUDENDORFF (H.). Handbuch der Astrophysik. Band 4. Berlin, Springer, 1929. 501 pp.

ERRERA (J.). Polarisation diélectrique. Paris, Presses Universitaires de France, 1928. 172 pp.

EVE (A. S.) and KEYS (D. A.). Applied geophysics in the search for minerals. Cambridge, University Press, 1929. 10+253 pp.

FARADAY SOCIETY. Crystal structure and chemical constitution. A general discussion held by the Faraday Society, March 1929. London, Faraday Society, 1929.

FLETCHER (H.). Speech and hearing. London, Macmillan, 1929. 15+331 pp.

FOERSTER (M.). Grundzüge der Statik der Hochbaukonstruktionen. 2te ergänzte Auflage. (Repetitorium der Hochbaukonstruktionen, Heft 2.) Berlin, Springer, 1928. 6+155 pp.

FOKKER (A. D.). Relativiteitstheorie. Groningen, Noordhoff, 1929. 308 pp.

FOURNIER (F. E.). Cet opuscule présente l'intérêt nouvelle d'expliquer par quelles causes dynamiques un croiseur, notamment, utilise dans sa translation en mer calme, au profit de sa vitesse, une part de sa force motrice d'autant plus grande que son tonnage est moins grand. Paris, Gauthier-Villars, 1929.

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OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

BETHLEHEM, PENNSYLVANIA, ANNUAL MEETING, December 27-28, 1929.

Abstracts must be in the hands of the Secretary of the Society, Professor R. G. D. Richardson, 501 West 116th St., New York City, not later than November 30. A symposium on the differential equations of engineering has been arranged; invited addresses will be given by Professors H. W. March, Vannevar Bush, S. Timoshenko, Dr. T. H. Gronwall, Dr. A. Nadai, and Mr. R. H. Park; see page 740 of the September-October issue.

DES MOINES, IOWA, December 30-31, 1929.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, University of Wisconsin, Madison, Wisconsin, not later than November 30. There will be the usual joint session with Section A of the American Association for the Advancement of Science and with the Mathematical Association of America. Addresses will be given by Professor L. L. Dines and O. D. Kellogg; see page 740 of the September-October issue. The Josiah Willard Gibbs Lecture, by Professor Irving Fisher on *Applications of mathematics to the social sciences*, will be held on Tuesday, December 31.

NEW YORK CITY, February 21, 1930.

Abstracts must be in the hands of the Secretary of the Society, Professor R. G. D. Richardson, 501 West 116th St., New York City, not later than February 1. An invited address will be given by Mr. E. C. Molina on *The theory of probabilities.*

NEW YORK CITY, April 18-19, 1930.

Abstracts must be in the hands of the Secretary of the Society, Professor R. G. D. Richardson, 501 West 116th St., New York City, not later than March 28. The Committee on Program is arranging a symposium on Differential Geometry, in which Professors Oswald Veblen, L. P. Eisenhart, W. C. Graustein, and others will participate.

CHICAGO, April 18-19, 1930.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, University of Wisconsin, Madison, Wisconsin, not later than March 28. An invited address will be given by Professor Henry Blumberg on *Methods in point sets and the theory of real functions.*

R. G. D. RICHARDSON, *Secretary of the Society.*

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

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